The problem of negotiation in public procurements: An auction-theoretic analysis for the private values case[1]

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Abstract:

This article compares alternative mechanism designs for the procurement of public works or services. It analyzes the simple reverse auction with no reservation price, the reverse auction with public reservation price, with secret reservation price and, finally, with secret reservation price and the possibility of negotiation when the winning bid is above the reservation price. We make use of the framework of auction theory with symmetric, independent, and identically distributed private values and risk neutral participants. The theoretical analysis and the simulations suggest that setting a reservation price yields better results for the government. Moreover, it is more advantageous to announce the reservation price if this price is low, and more advantageous to keep it secret if that price is high. Ex post negotiation in the secret reserve price model induces an increase in participants' bids. However, if the social welfare that the public works or services create sufficiently exceeds its reservation cost, then the negotiation mechanism may be desirable.

Keywords:

Public procurement auctions; Public reservation price; Secret reservation price; Ex post negotiation in procurements.

JEL classification codes: C72, D04, D44, D47

1. Introduction

Most public acquisitions, from basic consumption goods to the contracting of very expensive infrastructure projects such as a hydroelectric plant, are carried out through public tenders. Auctions are regulated by specific laws, which are periodically revised to generate the best possible return to the public sector.

The most recent of these revisions in Brazil took place on April 1, 2021, when Law No. 14,133/2021 came into force, establishing general rules for bidding and contracting for direct, municipal and foundational administration. This law introduces innovations aimed at improving the economic efficiency of public procurements. One of the bidding formats included in the law provides for the determination of a maximum reserve value, beyond which the work or service will not be contracted, which may or may not be announced before bidders place their bids. If the reserve value is kept secret, there is still the possibility for the government to negotiate with the winning bidder a reduction in its price, if this lowest winning price is above the reserve value.

The rationale for ex post negotiation would be to prevent the bid from failing if the lowest winning price was above the secret reserve price.

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The final goal of this research is to evaluate the advantages of this negotiation instrument with the tools of auction, focusing on the incentives on bidders' strategies. To achieve this objective, the paper adopts a broader approach, analyzing four different mechanisms. First, it analyses the basic procurement auction model without any imposition of a reserve price, even if such a reference price exists. Second, it includes an announced reserve price. Third, it considers the secret reservation price model, in which there is no contracting if the winning bit is above the secret reservation price. Finally, under the secret reserve price framework, the fourth model gives the winning bidder the possibility to lower its bid to the reservation price in order to win the contract, in case its bid is above it.

All the mechanisms studied here follow the independent, private, symmetric, and identically distributed values framework with risk-neutral participants. For each of the four models studied, we calculate the symmetric Nash equilibrium and present simulations for the basic case in which the bidders' costs are uniformly distributed on the interval [0,1], in order to compare the different bidding formats' results.

The analysis of the simplest model without reservation prices suggests that it can lead to the contracting of a work or service whose social return is below its cost, with a negative net social return. However, the higher the number of participants, the lower the players' bids and therefore the lower the expected cost to the government, which increases the expected social return.

Next, we show that including a reserve price guarantees that there will be no negative net social return on the bidding. However, this advantage practically disappears as the number of participants increases, which reinforces the importance of competition.

Including a secret reserve price has a similar effect to adding one more bidder, which increases competition and therefore reduces equilibrium bids. However, the lower the reserve price, the more likely the secret reservation price procurement will fail. Therefore, if the reserve price is low, it is better for the government to make it public. On the other hand, simulations show that if this reserve price is high enough, the government's optimal decision is to keep it secret.

Finally, we allow for ex-post negotiation, i.e., if winner's bid is above the secret reserve price, then, the corresponding bidder can still lower his bid to the value of the reserve price and thus ensure contracting. In this case, we show that the possibility of negotiation reduces competition, causing bidders to increase their bids, which implies a higher expected cost for the government. Therefore, if the reserve value of the work reflects the resulting social welfare, then negotiation is unfavorable in the model of independent private values and should be avoided. On the other hand, this article shows that if the social welfare generated by contracting the work exceeds its reserve value, then as the reserve value increases, so does the competition in the bidding (given by the number of bidders), in which case negotiation is advantageous for the government.

The contribution of this research to the literature are threefold. As procurement theory is essentially a reinterpretation of auction theory, the equilibria found for the basic models without reserve price, with public reserve price and with secret reserve price can be seen as a translation of dual equilibria in auctions. However, perhaps for that very reason, research papers and academic texts carefully studying equilibria in the specific context of public procurement are not widely available. The first contribution of this paper is didactic, presenting this direct and detailed derivation of the equilibria. Although the interested reader can consult the references cited from auctions and even follow the script presented in De Castro and De Frutos (2010), for example, to do this translation on his own, it seemed convenient to present and derive directly the equilibrium procurement bid strategies in order to make this literature more accessible to a wider audience.

Second, the theory of auctions is not unanimous as to the comparison between the use of public versus secret reserve prices, either identifying situations in which the former format generates greater returns to the auctioneer (e.g., Wilson and Weber 1982), or situations in which the latter format is more advantageous (e.g., Brisset and Naegelen 2006). Even more relevant for applied questions, auction theory makes this comparison between the two formats from the perspective that the reserve price, whether public or private, can be chosen strategically, aiming to maximize the auctioneer's expected revenue. However, in public tenders, according to the law, the reserve price is obtained through cost estimates and cannot be chosen strategically. The second contribution of this research is to compare the two bidding formats without the assumption that reserve price is chosen optimally.

Finally, the third contribution of this research is to offer a theoretical assessment of the effect of ex-post negotiation —when there is no bid less than or equal to the reserve value— on the expected cost to the government. Although this format is proposed in real-world situation, such as the new Public Procurement Law in Brazil (Brasil 2021), the authors did not find a theoretical model evaluating the trade-offs of ex-post negotiation.

In addition to this introduction, the article is organized as follows. Section 2 discusses changes in recent legislation in Brazil regarding the allowed bidding formats, as well as reviews the existing literature on the subject. Section 3 deals with the model without reserve price; section 4 analyses the public reserve price bidding model; section 5 studies the secret reserve price; section 6 introduces ex post negotiation in the secret reserve price model and, finally, section 7 concludes and discusses future extensions to the research.

2. The ebb and flow of the secret reserve value and ex-post negotiation

Administrative law in Brazil underwent an important change recently with the enactment of Law No. 14,133/2021, which established general rules for procurement auctions and contracting for direct, autonomous, and foundational administration. Several rules referring to tenders and contracts were changed with this new rule, and it is possible to highlight innovations concerned with the economic efficiency of public contracts, such as the establishment of a new type of bidding (called the competitive dialogue), or the establishment of an additional selection criterion based on higher greater economic return (to promote efficient contracts), or the obligation to present a matrix of risks and responsibilities between the parties (characterizing the initial economic-financial equilibrium of the contract in view of supervening events).

Law No. 13,303/2016, known as the “State-Owned Companies Law”, also provided for that selection criterion and the risk matrix requirement, so that the new bidding law ended up expanding such institutes to the entire Public Administration. In line with these changes, the profession expected that the principle of secrecy of the reservation price, first introduced by Law No. 12.462/2011 (which proposed a "Differentiated Contracting Regime" (RDC) to accelerate preparatory works for the World Cup), would also be extended to the direct, autarchic, and foundational administration[4].

In contrast to the public procurement law and previous contracts, Law No. 8.666/1993, which provided for the full publicity of the elements of the bidding and the contract[5], the RDC Law and the State-Owned Companies Law provided that the standard to be followed by the Administration would be the confidentiality of the estimate of the value of the contract, using very similar texts in this sense[6].

However, the recently sanctioned legal text promoted a reversal of that expectation, as the estimate of the value of the contract can only be confidential if there is justification for doing so[7].

An interesting aspect of this reversal can be observed in the process of sanctioning the law proposed by Congress. There was a veto of the item that established that the secret budget would be made public only and immediately after the proposal judgment phase. This veto ends up reinforcing a feeling that budget secrecy is not really desirable in light of the new law, since the moment of publicizing the estimate is not restricted.

Several questions emerge: was there a prevalence of the idea of publicity of administrative acts? Could it be that some hiring experiences were decisive for this, such as, for example, some frustration about the results obtained with secrecy? Has it been considered some

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[4] This law was initially intended for large-scale works to enable the 2016 Olympic and Paralympic Games, the Confederations Cup of the International Association of Football Association (FIFA) 2013, the FIFA World Cup 2014 and infrastructure works for airports. Over time, it also started to deal with actions of the Growth Acceleration Program (PAC), works within the scope of the Unified Health System (SUS), works and engineering services for construction, expansion and renovation and administration of penal establishments. and socio-educational service units, as well as engineering works and services related to improvements in urban mobility or expansion of logistics infrastructure.

[5] Art. 40. The public notice shall contain in the preamble the serial number in the annual series, the name of the interested department and its sector, the modality, the execution regime and the type of the bidding, the mention that it will be governed by this Law, the place , date and time for receiving the documentation and proposal, as well as for opening the envelopes, and must indicate the following: (...) § 2 Attachments to the public notice form an integral part of it: (...) II - estimated budget in spreadsheets of quantitative and unit prices; (Wording provided by Law No. 8,883 of 1994).

[6] Law No. 12.462/2011: Art. 6 Subject to the provisions of § 3, the budget previously estimated for the contracting will be made public only and immediately after the closing of the bidding, without prejudice to the disclosure of the details of the quantities and other information necessary for the preparation of the proposals. Law No. 13.303/2016: Art. 34. The estimated value of the contract to be entered into by the public company or by the mixed capital company will be confidential, being made available to the contracting party, upon justification in the preparation phase provided for in item I of art. 51 of this Law, give publicity to the estimated value of the object of the bidding, without prejudice to the disclosure of the details of the quantities and other information necessary for the preparation of the proposals.

[7] Law No. 14.133/2021: Art. 24. Provided that it is justified, the estimated budget for the contract may be confidential, without prejudice to the disclosure of the details of the amounts and other information necessary for the preparation of the proposals, and, in this case: I - confidentiality will not prevail for the internal and external control; II - (VETOED).

economic theory that already points to the disadvantage of secrecy in bids? Were there operational difficulties in undertaking secrecy? Or yet, did the conjunction of these factors discourage the use of secrecy?

The access to information law, Law nº 12.527/2011, can be a relevant factor in this context, as it established elements for the transparency of government actions as the basis of good governance, expressly indicating that secrecy should be an exception[8].

Regarding the frustration about the rule on secrecy, an emblematic example is in Souza (2013), in relation to the use of the RDC by Infraero, when he compared bids undertaken before and after the use of this form of contract, in which used secrecy in budgets. The conclusion was that there was an important reduction in the level of discounts of the winning companies in relation to the price budgeted by Infraero when compared to bids undertaken in previous moments with public budgets, resulting in more expensive contracts under the aegis of secrecy. In this way, what was achieved was in the opposite direction of what was expected with the establishment of secrecy, since there was an expectation of improving the advantages of contracts for the Administration.

This expectation was built on an economic outlook that coincided with an OECD recommendation[9]. For that organization, the secret budget is indicated as a way to minimize the damages resulting from the cartelized behavior of companies and the practice of raising prices, as pointed out by Rezende (2011). It was also expected that there would be incentives for the bidder to “race to the bottom”, in a sort of “winner’s curse” resulting from the dispute, as pointed out by Nóbrega (2015).

However, in the opposite direction, Rezende (2011) already indicated that the degree of corruption in society could not allow the expected mitigating effect on cartelized behavior, insofar as cartel members could obtain privileged information from corrupt public agents.

Along these lines, Nóbrega (2015) also already indicated disadvantages of secrecy, since the “dive” in prices could increase the risk of adverse selection of contracted companies that offered prices eventually lower than their production costs, requiring subsequent contractual rebalancing or abandonment. of contracts. This author also alerted to the fact that, as market prices are reasonably known by bidders, the advantages of secrecy obtained in studies of different types of auctions are very fragile, and it is not possible to conclude that this institute is adequate from the point of view of strictly economic.

Theoretical studies based on the theory of auctions have been developed to analyze the condition of secrecy of the auctioneer's reserve price, which, in the context of public bidding, represents keeping the value estimated by the Administration in secrecy. These studies included simplifications to allow the mathematical solution of the models, and different conclusions were obtained about the secrecy of the reserve prices of the auctions.

[8] Art. 3 The procedures provided for in this Law are intended to ensure the fundamental right of access to information and must be carried out in accordance with the basic principles of public administration and with the following guidelines: I - observance of publicity as a general precept and secrecy as an exception.

[9] As suggested in “Guidelines for fighting bid rigging in public procurement”, available at https://www.oecd.org/competition/cartels/42851044.pdf. Accessed on 04/08/2021.

Riley e Samuelson (1981) argumentam que, quando os participantes são neutros ao risco e há simetria e independência entre as valorações desses participantes, manter o preço de reserva secreto não aumentaria a receita esperada do leiloeiro, desde que o preço de reserva público seja escolhido estrategicamente para maximizar a receita do leiloeiro.

Milgrom and Weber (1982) also stated that revealing information, including the reserve price itself, would always be better for the auctioneer than keeping it secret in the so-called affiliation models, in which there would be no independence from the valuations of the various participants, that is, when the value of the object for one participant is somehow related to the values of that object for the other competitors. Elyakime et al. (1994) also found that the public reserve price would be better than the secret price for the purposes of the auctioneer's expected revenue, distinguishing the auctioneers' strategies from those of the participants, who would tend to bid lower than their valuations. In all these studies, the reserve price is chosen strategically, which makes it impossible to directly apply these results in the context of the public sector, where the law does not allow the manager to strategically choose the reserve price in order to optimize its expected benefit.

Conversely, even maintaining the possibility of strategic choice of reserve value, Vincent (1995), predicting auctions with common values instead of private ones, showed that keeping the reserve price secret could increase the auctioneer's revenue by inducing greater participation. of interested parties in the auction, in the case of the second-price auction, in which the winner pays the amount announced by the runner-up in the auction. Also Brisset and Naegelen (2006), considering risk-averse participants with private and independent values, concluded that if the relative risk aversion coefficient is constant and sufficiently high, the policy of keeping the reserve price secret would be optimal.

Finally, Silva (2011) concludes that reserve price secrecy could provide greater expected revenue to the auctioneer from a modeling with probability distribution functions in meshes, to allow addressing the interdependence of participants' valuations in an alternative way to that undertaken by Milgrom and Weber (1982). The result obtained derives from the fact that the secrecy of the reserve price causes uncertainty about its value, which promotes higher bids by auction participants, a result that would be affected by the level of the reserve price.

These studies focus on frictions in the theoretical model that would affect bidding results. But there are also several simple and operational issues that have the potential to affect the legislator's decision regarding whether or not to secrecy of bid value estimates.

A typical operational difficulty is to effectively maintain the secrecy of budgets for large enterprises whose budget allocations are contained in annual budget laws or multi-annual plans, or for works whose construction pattern or unrestricted use of government reference systems already make it possible to know, with great approximation, the values involved.[10]

There was also an operational difficulty arising from keeping the reserve price in secrecy

[10] Such as SICRO and SINAPI, made available, respectively, by the National Department of Transport Infrastructure (DNIT) and Caixa Econômica Federal (CEF).

regarding the negotiation phase with the companies after the presentation of the respective proposals, given the premise that the contracting must not exceed the value estimated by the Administration and the legal provision that the estimate could only be made public immediately after the closing of the bidding. To overcome this obstacle in the negotiation, the jurisprudence of the Federal Court of Auditors (TCU) had to establish the possibility of making the budget public already in the negotiation phase, expanding the possibility of using this institute.

The theoretical literature on negotiation in sales mechanisms typically models negotiation as an alternative mechanism to the auction, and not a complementary one, in which the owner of the good to be sold follows a well-defined negotiation protocol (negotiated tenders or even framework agreements), such as, for example, negotiating with one stakeholder, then with another, and so on[11], or even pre-selecting a smaller group of stakeholders and only then initiating a competitive sales mechanism. Papers in this line of research typically seek to compare trading and auction mechanisms to determine which is better under some criteria, such as efficiency or return to the auctioneer (Bullow and Kleperer 1996). Therefore, they do not include the type of question that interests us here.

Alternatively, there is literature on two-stage auctions. However, this literature seeks to design optimal sales mechanisms in multidimensional contexts (bids) or sequential auctions, nor does it address the issue of negotiation to match the reserve price, which is our interest.[12].

There is also the more recent literature known as negotiauction, which tries to include negotiation and auction phases in the same mechanism (Subramanian, 2010). For example, Ivanova-Stenzel and Kroger (2005) study a model in which there is an initial negotiation between the seller and a buyer and, if this negotiation does not lead to an agreement, there is a traditional auction involving a second interested buyer. Generally speaking, in a negotiauction there can be negotiation phases followed by auction phases, followed by new negotiation phases, etc. The general idea is that, by introducing negotiation phases into the sales process, a greater return can be obtained for the seller. This approach applies most naturally to the context of complex, multidimensional objects, where certain features of the object are valued by some buyers, while other buyers value other features of the object. Perhaps for this reason, they tend to be more used in online sales (Pham, 2013; Teich et al., 2001). In this context, it can be said that the negotiation after the conclusion of the bidding phase in a bidding fits in the negotiauctions literature. However, perhaps because this literature is broader, the authors did not find any academic article that modeled the specific negotiation mechanism being provided for in the new public procurement law.

All these factors end up pointing out the importance of understanding the economic mechanisms involved in public bids in order to study the advantage of providing in Brazilian legislation the possibility of secrecy of estimates of the values of public contracts and the prerogative of negotiation with the bidder winner after the bidding phase

[11] The basic reference in that literature is Bullow e Kleperer (1996). See also Kersten et al. (2016), Kirkegaard (2004) and Albano and Sparro (2008; 2010).

[12] See, for example, Branco (1997) and Katzman (1999).

is complete. What is expected is to identify in which situations such institutes can be properly used, since they are discretionary elements attributed to the Public Administration with the advent of the New Law of Bidding and Contracts. This is the main objective of the present work.

3. Procurement auctions with no reserve prices

The basic model

We start the theoretical modeling assuming that the government does not impose any reserve value (maximum) on the good/service being purchased.

The government wants to acquire a good or service through a public bidding process, in which n bidders participate. In what follows, we will use the term "work" indiscriminately to represent the good or service being purchased by the government.

Each bidder $i = 1,2,\dots,n$ incurs cost X_i to complete the work, so that X_i is the smallest value i is willing to accept to do the work. Value X_i is modeled as a random variable é distributed on the interval $\Omega = [\underline{\omega}_i, \overline{\omega}_i] \in \mathbb{R}_+$ with probability distribution function $F_i(\cdot)$. Distribution $F_i(\cdot)$ is interpreted as a description of all possible costs that bidder i may have to incur in order to do the work before the bidder learns the complete specifications of the work, i.e., in an ex-ante perspective. Distribution $F_i(\cdot)$ is also what the other participants $j \neq i$ believe about the costs of bidder i, since they play an incomplete information game.

The independent values model assumes that each participant has its own probability distribution $F_i(\cdot)$ and it is statistically independent of the other players distributions.

The private values model assumes, in addition, that each bidder observes its own value, but is not able to acquire any information regarding the values of the other participants except the ex-ante distributions $F_i(\cdot),\ i = 1,\dots,n$.

The symmetric values model adds the (Bayesian) hypothesis that all participants are ex-ante identical, which corresponds to assuming that $\underline{\omega}_i = \underline{\omega}$, $\overline{\omega}_i = \overline{\omega}$ e $F_i = F,\ i = 1,\dots,n$. Let $\Omega = [\underline{\omega}, \overline{\omega}]$.

Note that the independence hypothesis assumes that each realization c_i of the random variable X_i is an independent draw of the same distribution. The model assumes, furthermore, that function F has a continuous probability density function $f = F'$ with full support, i.e., $f(x) > 0,\ \forall x \in \Omega$. Function F and the number of players n are common knowledge to all participants.

The procurement auction is organized as follows: each player delivers to the government a sealed envelope in which he writes the minimum amount he is willing to receive for the execution of the work, his bid, without observing the bids of the other players. The government opens the envelopes and procures the work to the player who has made the lowest bid, the winner. The government pays for the work that bid. If k players have made the lowest bid, then a draw is held in which each of these players is selected the winner with the same probability $1/k$[13].

The rules of the procurement tender yield a Bayesian game between the n bidders, $\mathcal{J} = \left(n,\ (T_i)_{i=1,\dots,n},\ p, (A_i)_{i=1,\dots,n}, (u_i)_{i=1,\dots,n}\right)$ where, for all $i = 1, \dots, n$: (i) $T_i = \left[\underline{\omega}, \overline{\omega}\right] = C_i$ is the set of player i's types, i.e., the set of costs player i may incur to provide the work; (ii) $p(x_1, x_2, \dots, x_n) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n)$ is the joint probability density function; (iii) $A_i = \left[\underline{\omega}, \overline{\omega}\right] = L_i$ is the set of bids of player i; (iv) $u_i: L_i \times L_{-i} \times T_i \to \mathbb{R}$ is the payoff function of bidder i.

Let $l = (l_1, \dots, l_n)$ be a *ex post* bid profile of players. Then the ex post payoff of player i with cost c_i is:

$$u_i(l; c_i) = \begin{cases} l_i - c_i & \text{if} \quad l_i < \min_{j \neq i} l_j \\ 0 & \text{if} \quad l_i > \min_{j \neq i} l_j \\ \frac{l_i - c_i}{\left|\{k \mid l_k = \min_{j \neq i} l_j\}\right|} & \text{if} \quad l_i = \min_{j \neq i} l_j \end{cases}$$

A strategy profile of this game is a profile of n functions $\lambda = (\lambda_1, \dots, \lambda_n)$ where, for each $i = 1, \dots, n, \quad \lambda_i: \begin{cases} T_i & \longrightarrow & L_i \\ x_i & \longmapsto & l_i = \lambda_i(x_i) \end{cases}$.

The symmetric equilibrium

The following proposition calculates the symmetrical equilibrium of this game. The proof of this proposition, as well as of all other propositions and the details of all the calculations developed here, is available in the Appendix. In Proposition 1, as in the following propositions, we will repeatedly use the distribution $G(x) = 1 - [1 - F(x)]^{n-1}$, as well as its density function $g(x) = G'(x) = (n-1)[1 - F(x)]^{n-2} f(x)$. Therefore, to avoid repetitions, in what follows we will not specify these functions.

Proposition 1. *In the symmetric, strictly increasing, differentiable equilibrium of the procurement auction with* ***no reserve price****, each player chooses the bid strategy* λ^{spr} *given below.*

$$\lambda^{spr}(x) = \frac{1}{1 - G(x)} \int_x^{\overline{\omega}} y g(y) dy = x + \frac{1}{1 - G(x)} \int_x^{\overline{\omega}} [1 - G(y)] dy$$

Example 1. Suppose the values are uniformly distributed over the interval [0,1]. Then, a player of type x bid is $\lambda^{spr}(x) = \frac{n-1}{n} x + \frac{1}{n}$.

Therefore, a player's bid is a weighted average of its true cost (with weight $\frac{n-1}{n}$) and the maximum possible cost.

[13] It is worth noting that, as we are dealing with continuous distributions, we will seek strictly increasing Nash Bayesian equilibria, that is, in which the bids of the participants will be strictly increasing in their values. In this case, the probability of a tie occurring is zero.

In particular $\lim_{n\to\infty} \lambda^{spr}(x) = x$, i.e., as competition increases, the player's bid moves closer and closer to his own cost of supplying the work.
When $n = 2$, $\lambda^{spr}(x) = \frac{x+1}{2}$, which is the expected value of the second lowest cost, conditional on x being the lowest cost.

Proposition 2. *The expected payment from the government in a sealed lowest cost procurement auction with no reserve price is* $m^{spr} = n\int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy$

Example 2. Consider again the special case where the values are uniformly distributed over [0,1]. Then $m^{spr} = n\int_0^1 F(y)yg(y)dy = \frac{2}{n+1}$

In particular, when $n = 2$, the expected payment is 2/3. Furthermore, as the number of participants increases, the payment decreases, since greater competition causes bidders to demand lower remuneration for their services.

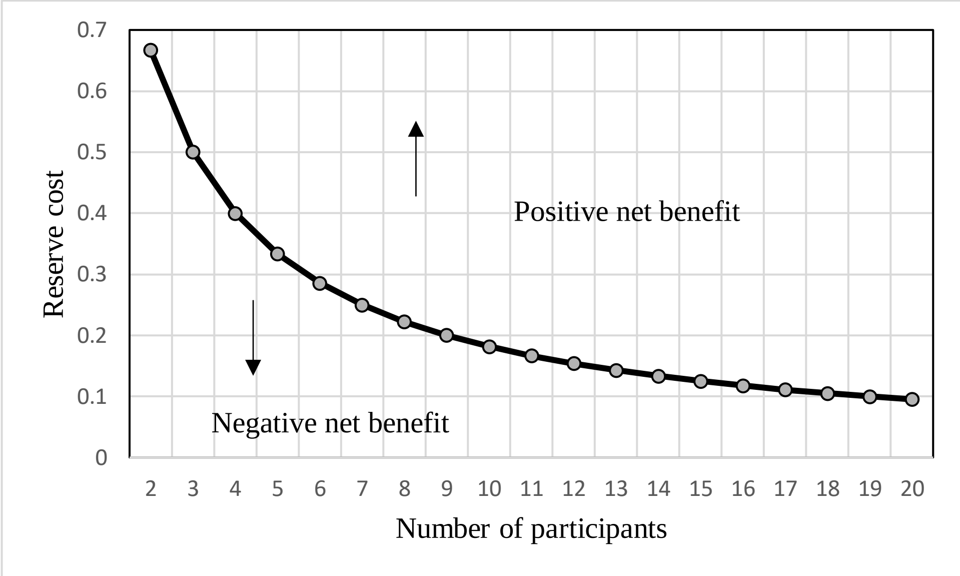
Suppose the government is willing to pay, at most, the amount r for the work. This value reflects the cost estimates of the work at market prices made by the bodies responsible for the bidding. The amount r is called the reservation cost or reservation price of the work and is seen in the present study as the (gross) social benefit of the project. Then, the net expected social benefit is $r - m^{spr} = r - n\int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy$.

Note that depending on the value of r, not setting a reserve price in the bidding design can lead to a negative net expected benefit. In fact, as the work is contracted with probability 1, this occurs if $r < n\int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy$.

Graph 1 below presents the relationship between the reservation cost and the number of participants. The smaller the number of participants, the higher the reservation cost will have to be so that the competition (without including the rule of not contracting the work if the winner bids higher than the reservation cost) does not lead to an expected net loss.

For instance, with two participants, the reservation cost (i.e. the value of the work to the government) must be above 66% of the maximum possible value of the cost of the work, that is, the upper limit $\overline{\omega}$ of the range of possible costs, so that no losses are incurred. On the other hand, as competition increases, the situations in which bidding without reserve cost induces expected net loss are reduced. With 20 participants, then a reservation cost above only 10% of the maximum possible value of the cost of execution of the work for the government ensures no expected net loss. The table with the exact values, as well as all the other simulations made in this work are available on demand to the authors.

Graph 1. Number of participants and reserve cost



Note: The curve represents the minimum reserve cost (i.e. the minimum return of the work to the government) so that the auction without reserve value requirement does not generate a negative expected net benefit.
Source: Authors' calculations.

4. Procurement auctions with public reserve prices

We now consider the same model studied earlier, in which the following restriction is included: the government announces a maximum reserve price, r, so that, if the winner of the bid has bid greater than r, i.e., if the lowest payment requested is greater than r, then the work will not be contracted.

The symmetric equilibrium

Proposition 3. *In the symmetric, strictly increasing, differentiable equilibrium of the procurement auction with* **public reserve price***, each player chooses the bid strategy* λ^{pra} *given below.*

$$\lambda^{pra}(x) = x \qquad if \qquad x \geq r$$

$$\lambda^{pra}(x) = r\frac{1 - G(r)}{1 - G(x)} + \frac{1}{1 - G(x)}\int_x^r yg(y)dy \qquad if \qquad x \leq r$$

Example 3. Suppose the values are uniformly distributed over the interval [0,1]. Then, a player of type x bid is $\lambda^{pra}(x) = \frac{n-1}{n}x + \frac{1}{n} - \frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$

Note that if $r = 1$, then $\lambda^{pra}(x) = \frac{n-1}{n}x + \frac{1}{n}$ which is precisely the solution with no reserve price.

Proposition 4. *The expected payment from the government in a sealed lowest cost procurement auction with public reserve price r is:*

$$m^{pra}(r) = nr\big(1 - G(r)\big)F(r) + n\int_{\underline{\omega}}^{r} F(y)yg(y)dy$$

Corollary. The bid of a player that has value x greater than the announced reserve value $r < 1$, $\lambda(r)$ is a strictly increasing function of r. Furthermore, the government's expected payment, $m^{pra}(r)$, is also a strictly increasing function of the maximum reserve price r.

Note. It is worth noting that the corollary suggests choosing a very low reserve price to lower the cost of the contract to the government. However, this will not be optimal as there is a trade-off: On one hand, a low reserve price r induces bidders who have costs below r to lower their bids; but, on the other hand, those bidders who have costs above will not be able to bid competitively, increasing the probability of auction failure, in which case the government has zero net benefit. It is possible to show (see Laffont and Maskin 1980) that if it were possible to choose the reserve price strategically, the government's expected return would be maximized with a value strictly lower than the real social benefit of the work, although not too low due to the exclusion of participants. of higher cost. However, unfortunately, in the case of the government, the legislation prevents the public agent from choosing the reserve value strategically.

Example 4. Suppose the values are uniformly distributed over the interval [0,1]. Then,

$$m^{pra}(r) = \frac{2}{n+1}[1 - (1-r)^n(nr+1)]$$

Note that

$$m^{pra} = \frac{2}{n+1} - \frac{2}{n+1}(1-r)^n(nr+1) < \frac{2}{n+1} = m^{spr}$$

When $n = 2$:

$$m^{pra} = \frac{2}{3} - \frac{2}{3}(1-r)^2[2r+1] < \frac{2}{3} = m^{spr}$$

Proposition 5. *The government's expected payment in a lowest cost sealed bid procurement auction with an announced reserve price $r > \underline{\omega}$ is less than the expected payment in a lowest cost sealed bid auction with no reserve price: $m^{pra}(r) < m^{spr}$.*

Note. It is noteworthy that, as the expected payment is lower with the inclusion of the reserve price constraint, and as the payment only occurs in advantageous situations for the government, that is, when the benefit is greater than the price of reserve, then it is immediate to conclude that the net benefit is greater with the reserve price requirement.

More precisely, in the current model, with the reserve price corresponding to the estimated market cost, r, the work will only be contracted if the lowest cost of the participants is less than r, since, in equilibrium if $x \leq r$ then, $\lambda(x) \leq r$. Therefore, the work is undertaken with probability:

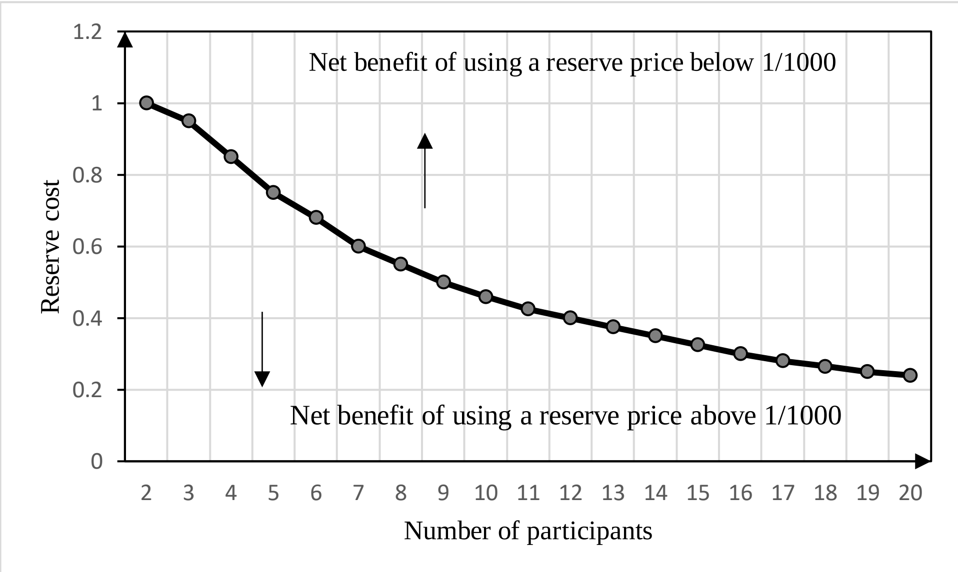
$$\text{Prob}\left[\min_{i=1,\ldots,n} x_i \leq r\right] = Prob[x_1 \leq r \vee \ldots \vee x_n \leq r]$$

$$= 1 - Prob[x_1 > r \wedge \ldots \wedge x_n > r] = 1 - \left(1 - F(r)\right)^n$$

Hence, the expected return to the government is: $r\left[1 - \left(1 - F(r)\right)^n\right] - m^{pra}(r) = r\left[1 - \left(1 - F(r)\right)^n\right] - nr\left(1 - G(r)\right)F(r) - n\int_{\underline{\omega}}^{r} F(y)yg(y)dy$

However, as competition increases, the benefit of including reserve prices reduces greatly. Graph 2 presents the limit values of the reserve cost from which the difference between the net benefit of the two bidding formats is less than one thousandth of the maximum possible value of the cost of the work. It is noted that, when competition is small, there is greater benefit in announcing a maximum reserve cost and not contracting the work if the highest bid is greater than this cost. For example, if there are only 4 participants, the reserve cost must be greater than 80% of the maximum possible value of construction of the work so that there is no significant advantage in using the reserve cost; on the other hand, if there are 10 participants, a reverse value around 50% of the maximum value is enough for the two formats to be essentially equivalent for the government.

Graph 2. Difference between the net benefit when the announced reserve cost is used and when there is no maximum cost requirement



Note: The curve represents the minimum reserve cost for the auction with a reserve value requirement from which the introduction of a reserve price entails a negligible increase (less than one thousandth of the maximum possible value of the cost of executing the work) of the net benefit of the government, compared to the simple auction with no reserve price.

Source: Authors' calculations.

5. Procurement auctions with secret reserve prices and no negotiation

Suppose now that there is a reserve value r for the work, but that this value is only revealed after bid are made. In this case, if the winning bid (the lowest of the bids) is above the reserve value, the work will not be contracted.

From the bidders' point of view, the reserve value is a random variable r distributed in Ω according to a probability distribution H and respective density h.

The symmetric equilibrium

Proposition 6. *In the symmetric, strictly increasing, differentiable equilibrium of the procurement auction with* ***secret reserve price****, each player chooses the bid strategy* λ^{prs}*given below.*

$$\lambda^{prs}(x) = x + \frac{1}{1 - G(x)} \int_x^{\overline{\omega}} \frac{1 - H\big(\lambda^{prs}(y)\big)}{1 - H\big(\lambda^{prs}(x)\big)} [1 - G(y)] dy$$

Corollary. *If the ex-ante distribution of the reserve price,* H*, is strictly increasing, then the introduction of a secret reserve price reduces the bidder's bids, compared to the situation where there is no reserve price. Therefore, the use of a secret reserve price reduces the government's expected payment.*

Note. It is worth noting that the expression in Proposition 6, unlike the expressions in Propositions 1 and 3, does not present an explicit form for the solution of the problem, since $\lambda^{prs}(x)$ appears on both sides of the equation. Therefore, without specific knowledge of the probability distributions, it is not possible to isolate the solution.

In order to obtain an explicit expression for the solution of the problem, we particularize for the case of uniform distributions and look for linear solutions.

Example 5. Consider again the special case where the participants' costs are uniformly distributed in the interval [0,1], as well as the reserve price r. Then:

$$\lambda^{prs}(x) = \frac{n}{n+1}x + \frac{1}{n+1}$$

Note. Comparing the expressions $\lambda^{prs}(x) = \frac{n}{n+1}x + \frac{1}{n+1}$ with, $\lambda^{spr}(x) = \frac{n-1}{n}x + \frac{1}{n}$, we note that the secret reserve price works as if the players were facing one more competitor (the government itself) so that their bids become more aggressive (lower), which benefits the government.

Comparing with the public reserve price case, $\lambda^{pra}(x) = \frac{n-1}{n}x + \frac{1}{n} - \frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$ we have this same effect that favors the secret (it works as if it increases competition), but also an effect that favors the announced model, the negative term $-\frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$. Which of the two effects will prevail is unclear and will depend on various parameters. In particular, if r is high, i.e. the cost constraint is not very relevant, then the term $\frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$ will be low, so so using a secret reserve price is expected to be advantageous. On the other hand, if r is low, that is, the government restricts the cost of the work quite a lot, then the term $\frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$ will be high, so the public reserve price model is expected to be desirable.

The expected government payment

Proposition 7. *The expected payment from the government in a sealed lowest cost procurement auction with secret reserve price* r *is zero (the auction fails) if* $\lambda(\underline{\omega}) > r$ *and if* $\lambda(\underline{\omega}) \leq r$ *is given by the expression below, where* $\lambda(y) = \lambda^{prs}(y)$ *is the bid of a participant of cost* y *in the Nash equilibrium found in Proposition 6.*

$$m^{prs}(r) = n\int_{\underline{\omega}}^{\lambda^{-1}(r)} x[1-G(x)]f(x)dx$$

$$+\, n\int_{\underline{\omega}}^{\lambda^{-1}(r)}\int_{x}^{\overline{\omega}}\left[1-H\big(\lambda(y)\big)\right][1-G(y)]dy\frac{1}{1-H\big(\lambda(x)\big)}f(x)dx$$

Example 6. Consider again the special case where the participants' costs are uniformly distributed in the interval [0,1], as well as the reserve price r and a linear solution. Then:

$$m^{prs}(r) = \frac{2n+1}{(n+1)^2} - \frac{(n+1)^{n-1}}{n^n}[nr+1](1-r)^n$$

When $n = 2$, if $\lambda(0) = \frac{1}{3} \leq r$, then $m^{prs}(r) = \frac{5}{9} - \frac{3}{4}[2r+1](1-r)^2$

In that case, the secret reserve price reduces government expected payments (in comparison to the announced reserve price case) because it increases the risk a bidder may lose if he chooses higher bids.

The government's net benefit

Assume as before that the work, once carried out, generates a social welfare gain $r \in \Omega = [\underline{\omega}, \overline{\omega}] \in \mathbb{R}_+$. In the current model, the work will only be contracted if the lowest bid of the participants is less than r, that is, the work will be executed with probability:

$$\text{Prob}\left[\min_{i=1,\dots,n}\lambda(x_i) \leq r\right] = \text{Prob}[x_1 \leq \lambda^{-1}(r) \vee \dots \vee x_n \leq \lambda^{-1}(r)]$$

$$= 1 - \text{Prob}[x_1 > \lambda^{-1}(r) \wedge \dots \wedge x_n > \lambda^{-1}(r)] = 1 - \left(1 - F(\lambda^{-1}(r))\right)^n$$

Hence, the government's net benefit from the work is:

$$r\left[1-\left(1-F\left(\lambda^{-1}(r)\right)\right)^{n}\right]-m^{prs}(r)=$$

$$r\left[1-\left(1-F\left(\lambda^{-1}(r)\right)\right)^{n}\right]-n\int_{\underline{\omega}}^{\lambda^{-1}(r)}x[1-G(x)]f(x)dx$$

$$-n\int_{\underline{\omega}}^{\lambda^{-1}(r)}\int_{x}^{\overline{\omega}}[1-H(\lambda(y))][1-G(y)]dy\frac{1}{1-H(\lambda(x))}f(x)dx$$

Comparing with the no reservation price model, including a secret reserve price has two advantages for the government. First, participants play more aggressively, as discussed earlier. Second, the work is only contracted when it generates a non-negative net benefit. Hence, naturally, the inclusion of the secret reserve price leads to a greater benefit to the government.

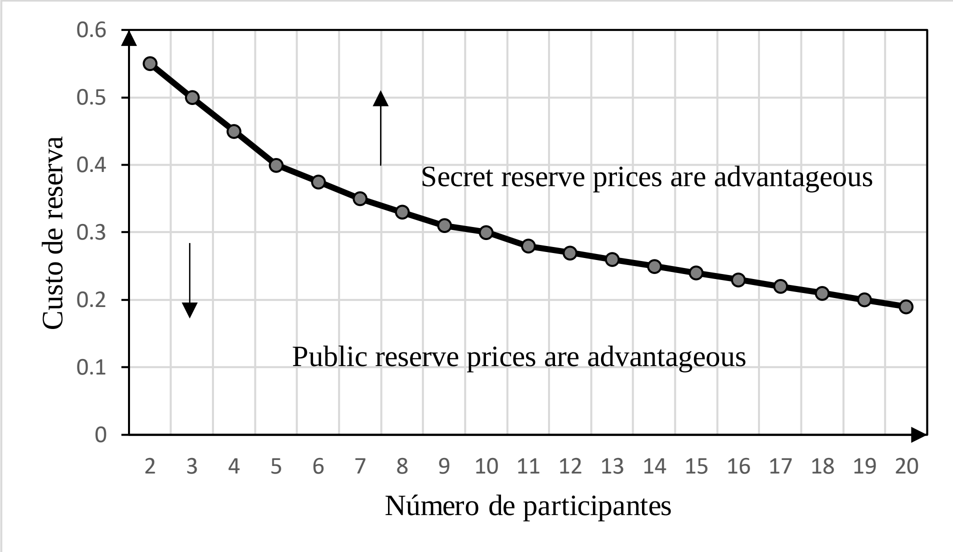
The comparison with the public reserve price model reveals that either format can be more advantageous for the government, depending, on the one hand, on the competition level of the auction, i.e., the number of participants and, on the other hand, on the other hand, the gross benefit of the work to society, r. Graph 3 presents the values of the benefit of the work r from which it is advantageous to keep the reserve price secret, depending on the number of participants.

In general, a public reserve price is more advantageous when it comes to low prices, that is, cheap works, with low production costs. On the other hand, the secret reserve price model proves to be more advantageous if this price is high, that is, for larger, higher-cost works. This result is similar to that found in Silva (2011) for the analysis of auctions. Furthermore, the greater the competition, that is, the greater the number of bidders, the greater the region for reserve price values in which the secret reserve price model is more advantageous.

It is also worth noting another role of competition, already highlighted in the previous comparison between the model without a reserve price and the model with an announced price. For $n=3$ participants, for example, it is only advantageous to keep the price secret if the social benefit of the work is above 50% of its maximum production cost. If it is lower, it is better for the government to publicly announce the reserve value. For a higher level of competition, for example with 10 participants, then it is enough that the social return is greater than 30% of the maximum value of the cost of execution of the work for it to be advantageous for the auctioneer to keep the reserve price secret.

The intuition behind this result is simple. There is a trade-off between keeping the price secret or revealing it publicly. On the one hand, secret reserve prices increase competition among participants, causing them to choose lower bids. On the other hand, it increases the probability that the work will not be contracted. If the social benefit is high, the reserve price is less limiting for contracting the work and, therefore, must be kept secret. On the other hand, a low social benefit implies a low probability of contracting the work when the reserve value is kept secret. In that case, it's best to make it public to increase the likelihood that the auction will be successful.

Graph 3. Comparison between public and secret reserve prices



Note: The curve represents the minimum reserve cost (that is, the minimum return of the work to the government) so that the auction with a secret reserve value is more advantageous for the government than the auction with a public reserve value. As these are simulations, the values are approximate.
Source: Authors' calculations.

6. Procurement auctions with secret reserve prices and negotiation

Suppose now that there is a reserve price r for the work, but that that value is only revealed after delivery of the envelopes. Furthermore, if the winning bid is above the reserve price, the winner can still choose to "match" the reserve value and thus avoid losing the contract.

From the bidders' point of view, the reserve value is a random variable r distributed in Ω according to a probability distribution H and respective density h.

The symmetric equilibrium

Proposition 8. *In the symmetric, strictly increasing, differentiable equilibrium of the procurement auction with* ***secret reserve price, but with ex-post negotiation****, each player chooses the bid strategy* λ^{psn} *given below.*

$$\lambda^{psn}(x) = x + \mathcal{H}\left(\lambda^{psn}(x)\right) - \mathcal{H}(x) + \frac{1}{1-G(x)}\int_x^{\overline{\omega}} [1-H(y)][1-G(y)]dy$$

where $\mathcal{H}$ *is a primitive of function* H.

Example 7. Consider again the special case where the participants' costs are uniformly distributed in the interval [0,1], as well as the reserve price r and a linear solution. Then:

$$\lambda^{psn}(x) = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} x + \left(1 - \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

Hence, the player's bid is again a weighted average between its true cost (weight $\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$) and the maximum possible cost, which also converges to 1 as n increases.

In particular $\lim_{n\to\infty} \lambda^{prn}(x) = x$, that is, when competition increases, the player's bid gets closer and closer to his own cost of supplying the work.

Note. It is interesting to compare the solutions found for cases of secret reserve price with or without negotiation. We have:

$$\lambda^{prs}(x) = \frac{n}{n+1}x + \frac{1}{n+1}; \quad \lambda^{psn}(x) = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} x + \left(1 - \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

$$\frac{n}{n+1} > \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} \Leftrightarrow 1 > 0$$

Therefore, $\lambda^{prs}(x) < \lambda^{psn}(x)$, that is, the reserve price mechanism with negotiation raises bidds, which is bad for the government. This is because in the new model, the only thing that the player needs to do is beat his opponents, since, having done that, if his bid is greater than the reserve value, he will still have the option to reduce it and execute the work, if his cost is below the reserve cost.

However, it should be noted that the possibility of negotiation means that the object is sold even when all bids are above the reserve value, if the winner has a cost below this reserve value.

A comparison with the other models analyzed yields:

$$\lambda^{prs}(x) < \lambda^{spr}(x), \lambda^{psn}(x) \quad \text{and} \quad \lambda^{spr}(x) > \lambda^{pra}(x; r)$$

The expected government payment

Proposition 9. *The expected payment from the government in a sealed lowest cost procurement auction with secret reserve price r and negotiation is given below.*

(i) If $\lambda(\underline{\omega}) > r$, the expected payment is: $m^{psn}(r) = nr \int_0^r [1 - G(x)] f(x) dx$

(ii) If $\lambda(\underline{\omega}) \leq r$, the expected payment is given below, where $\lambda(y) = \lambda^{psn}(y)$ is the bid of a participant of cost y in the Nash equilibrium found in Proposition 8.

$$m^{psn}(r) =$$

$$n \int_{\underline{\omega}}^{\lambda^{-1}(r)} x[1 - G(x)] f(x) dx + n \int_{\underline{\omega}}^{\lambda^{-1}(r)} \int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)] dy\, f(x) dx$$

$$+n\int_{\underline{\omega}}^{\lambda^{-1}(r)}[\mathcal{H}(\lambda(x))-\mathcal{H}(x)][1-G(x)]f(x)dx+nr\int_{\lambda^{-1}(r)}^{r}[1-G(x)]f(x)dx$$

Example 8. Consider again the special case where the participants' costs are uniformly distributed in the interval [0,1], as well as the reserve price r and a linear solution. Then: $\lambda^{psn}(0)=\left(1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$ and:

(i) For $r<1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$: $\quad m^{psn}(r)=r(1-(1-r)^n)$

(ii) For $r\geq 1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$: $\quad m^{psn}(r)=nA(s)+nB(s)+nC(s)+nD(s)$

Where,

$$nA(s)=\frac{1}{n+1}[1-(1-s)^n(1+ns)];\quad nB(s)=\frac{n}{n+1}\frac{1}{n+2}[1-(1-s)^{n+2}]$$

$$nC(s)=\frac{1}{2}\left[\beta^2+\frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2-1)+2\alpha\beta\right]\right]$$

$$-\frac{1}{2}\left[a(s)+\frac{1}{n+1}\left[c(s)+\frac{2}{n+2}(\alpha^2-1)[1-s]\right][1-s]\right][1-s]^n$$

$$nD(s)=r[[1-s]^n-[1-r]^n]$$

$$s=1-\alpha^{-1}(1-r);\ \alpha=\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}},\quad \beta=1-\alpha;\ a(s)=(\alpha^2-1)s^2+2\alpha\beta s+\beta^2$$

$$c(s)=2(\alpha^2-1)s+2\alpha\beta$$

The government's net benefit

Assume as before that the work, once carried out, generates a social welfare gain $r\in\Omega=[\underline{\omega},\overline{\omega}]\in\mathbb{R}_+$. In the current model, with a secret reserve price, but with the possibility of negotiation, the work will be contracted whenever the lowest value of the participants is less than r, that is, the work will be executed with probability:

$$\text{Prob}\left[\min_{i=1,\dots,n}x_i\leq r\right]=1-\left(1-F(r)\right)^n$$

Therefore, the government's net benefit from the work is: $r\left[1-\left(1-F(r)\right)^n\right]-m^{psn}(r)$.

Table 1 presents the net expected government benefit for procurement auction with secret reserve price and with negotiation in the case of uniform distributions. The first column corresponds to the number of players while the second column corresponds to the limit value so that if the reserve value is less than that limit, then if the work is contracted, it will be done at a cost equal to the reserve value, so that the expected net benefit will be zero. All gray cells are in this situation. For example, with $n = 2$ players, if the reserve price is less than 0.423, then if the work is contracted, it will be contracted in the negotiation process, so that the winner will receive exactly r for the work. In comparison with the situation of a secret reserve price without negotiation, the region in which the expected net benefit is null increases, although the region in which the work is effectively contracted also increases. What happens here is that, while in the case without negotiation there was a null probability of the work being contracted at cost r, now this probability is positive, because whenever there is negotiation, if the winner has a cost below r, the work is contracted at the price r, which gives zero net benefit to the government.

Table 1. Government's net benefit with a procurement auction with secret reserve price and negotiation, for different amounts of the benefit of the work and the number of participants with uniform distributions

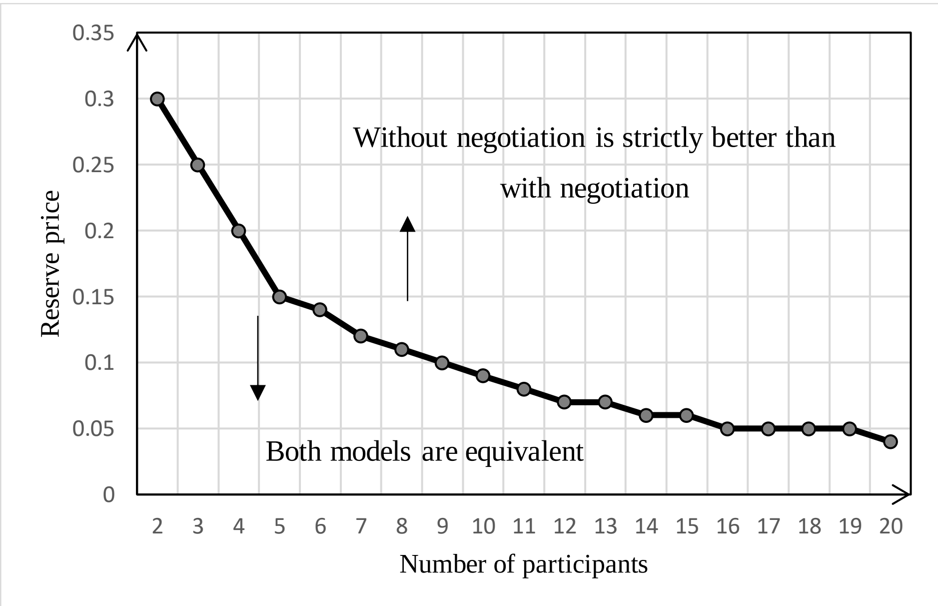
n	$\lambda(0)$	Reserve price r																		
		0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
2	0,423	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,010	0,026	0,049	0,078	0,112	0,151	0,193	0,238	0,286	0,335	0,385
3	0,293	0,000	0,000	0,000	0,000	0,000	0,007	0,022	0,045	0,075	0,109	0,148	0,191	0,236	0,283	0,331	0,381	0,430	0,480	0,530
4	0,225	0,000	0,000	0,000	0,002	0,013	0,034	0,063	0,098	0,137	0,180	0,225	0,273	0,321	0,370	0,420	0,470	0,520	0,570	0,620
5	0,184	0,000	0,000	0,000	0,012	0,034	0,065	0,102	0,143	0,188	0,234	0,282	0,331	0,381	0,431	0,480	0,530	0,580	0,630	0,680
6	0,155	0,000	0,000	0,007	0,027	0,057	0,094	0,135	0,180	0,227	0,276	0,325	0,375	0,425	0,474	0,524	0,574	0,624	0,674	0,724
7	0,134	0,000	0,001	0,015	0,042	0,077	0,119	0,164	0,211	0,259	0,308	0,358	0,408	0,458	0,508	0,558	0,608	0,658	0,708	0,758
8	0,118	0,000	0,004	0,025	0,057	0,096	0,140	0,187	0,235	0,285	0,334	0,384	0,434	0,484	0,534	0,584	0,634	0,684	0,734	0,784
9	0,106	0,000	0,009	0,034	0,070	0,113	0,159	0,207	0,256	0,305	0,355	0,405	0,455	0,505	0,555	0,605	0,655	0,705	0,755	0,805
10	0,095	0,000	0,014	0,044	0,083	0,127	0,174	0,223	0,273	0,322	0,372	0,422	0,472	0,522	0,572	0,622	0,672	0,722	0,772	0,822
11	0,087	0,001	0,019	0,052	0,094	0,140	0,188	0,237	0,287	0,337	0,387	0,437	0,487	0,537	0,587	0,637	0,687	0,737	0,787	0,837
12	0,080	0,002	0,024	0,061	0,104	0,151	0,200	0,249	0,299	0,349	0,399	0,449	0,499	0,549	0,599	0,649	0,699	0,749	0,799	0,849
13	0,074	0,004	0,030	0,068	0,113	0,161	0,210	0,260	0,310	0,360	0,410	0,460	0,510	0,560	0,610	0,660	0,710	0,760	0,810	0,860
14	0,069	0,006	0,035	0,075	0,121	0,170	0,219	0,269	0,319	0,369	0,419	0,469	0,519	0,569	0,619	0,669	0,719	0,769	0,819	0,869
15	0,065	0,008	0,040	0,082	0,129	0,178	0,227	0,277	0,327	0,377	0,427	0,477	0,527	0,577	0,627	0,677	0,727	0,777	0,827	0,877

Note: The gross benefit of the work is the reserve price, which is obtained only when the work is contracted. The parameter n corresponds to the number of players. This simulation is based on the assumption that bidders' cost values are uniformly distributed in the interval [0,1] as well as the reserve value. The bid function $\lambda^{psn}(x) = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} x + 1 - \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$ corresponding to the players' equilibrium bids when there are n participants is not shown in the table but is used in the calculations. The function $m^{psn}(r)$ corresponds to the government's expected expenditure, that is, the expected payment to the successful bidder for the execution of the work. The relevant variable is the net benefit: $r\left[1 - \left(1 - F(r)\right)^n\right] - m^{psn}(r)$. The parameter $\lambda(0)$ in the second column explains situations in which if the object is sold, it will be at its reserve value, generating a null net benefit.
Source: Authors' calculations.

Graph 4, in turn, shows the difference between net expected benefit secret reserve price, with negotiation and without negotiation. Note that negotiation is never to the government's advantage. As the bids are higher in the presence of negotiation, the works become more expensive and the introduction of negotiation becomes harmful, except in the particular case where the work is contracted at the reserve price. Note that in this case the work generates zero net social return, so that the two formats become equivalent. Note that even with low competition, e.g., $n = 3$, very low reserve values (above 25% of the maximum possible value of the cost of work for a bidder) guarantee the strict superiority of the no-negotiation model.

It is worth noting the important assumption made in this game that the reserve value is equal to the benefit generated by the work. This means that, every time the work is contracted due to the price reduction via negotiation, the net benefit for the government is null, as it pays exactly what the work is worth for itself. Therefore, the only advantage of negotiation, which would be to guarantee the contracting of the work in more situations than in the absence of it, is lost in this comparison. Therefore, if the modeling hypothesis according to which the reserve price corresponds to the value of the work for the government is adequate, there is no real justification for introducing the possibility of negotiation in the specific case of the private values framework. In summary, while the comparative analysis regarding secret versus public reserve price suggests the superiority of the secret reserve price when dealing with more expensive works, this section suggests that the secret reserve price without negotiation leads to a better result for the government than the use of ex post negotiation.

Graph 4. Comparison between secret reserve prices procurement auctions with and without negotiation



Note: The curve represents the minimum reserve cost (i.e., the minimum return of the work to the government) so that the auction with a secret reserve value without negotiation is strictly more advantageous for the government than the auction with a secret reserve value and negotiation. As this is a simulation, the values are approximate.
Source: Authors' calculations.

Social benefit above the reserve price: a possible advantage of the model with negotiation

This paper has shown the stylized fact that the availability of negotiation makes bidders choose higher bids, making the procurement auction less competitive and, consequently, more expensive for the government. The intuition behind this result is that, with negotiation, the concern about bidding too high and going above the reserve value disappears, as the winner can always reduce her bid ex post if it is to her advantage.

Furthermore, as the reserve value is seen as the social welfare of the work, whenever negotiation is used, there is no net advantage in the result, as the government pays for the work exactly the social benefit generated by it. However, one might think that there is a mismatch between the reserve value r, which is defined as the estimated competitive cost of the work, and its social benefit b, which is expected to be higher ($b > r$). If this is the case, then each time there is a negotiation, the benefit $b - r > 0$ must be added to the net welfare. But then the new net social benefit becomes:

$$b\left[1 - \left(1 - F(r)\right)^n\right] - m^{psn}(r)$$

Therefore, a trade-off between the two models emerges. If, on the one hand, in the negotiated model the government pays more for the work, in expected terms, it also increases the probability that the work will be carried out, generating an additional net social welfare gain. The following proposition establishes the relationship between b and r that must be satisfied for the negotiation model to be desirable.

Proposition 10. Let r be the reserve value corresponding to the production cost at competitive market values of the work and let $b > r$ be the social welfare gain obtained with the construction of the work. Then, bidding with reserve price and with negotiation will be advantageous to the government compared to bidding with reserve price but without negotiation if and only if $m^{prs}(r) = 0$ or $m^{prs}(r) \neq 0$ and:

$$b > \max\left\{r, b_{min} = \frac{\left(1 - F\left((\lambda^{prs})^{-1}(r)\right)\right)^n - \left(1 - F(r)\right)^n}{m^{prn}(r) - m^{prs}(r)}\right\}$$

Table 2 presents the minimum value that social welfare $b > r$ for it to be advantageous for the government to use the negotiation mechanism.

It is noteworthy that when the reserve value is very low, in which case the probability of contracting the work is zero in the absence of negotiation, any social welfare value greater than r makes the use of negotiation advantageous.

On the other hand, for intermediate values of r, where there is a positive probability that the work will be contracted, for example, between 30% and 60% of the maximum value of the work, then the social benefit must be significantly greater than r for negotiation to be desirable. This result is especially true when there is low competition. For example, when r is equal to 40% of the maximum value of the work and there are a maximum of

4 participants, then the social benefit needs to be greater than twice the highest possible cost of executing the work for it to be advantageous to use the negotiation mechanism.

Finally, as the reserve price increases, so does the probability of failure of the bid without negotiation and, therefore, the use of negotiation becomes advantageous again.

Table 2. Minimum social welfare b_{min} needed for the negotiation model to be advantageous to the government

n	$\lambda^{prs}(0)$	Reserve price r																		
		0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
2	0,333	0,000	0,000	0,000	0,000	0,000	0,000	2,851	2,414	2,010	1,693	1,438	1,223	1,032	0,855	0,684	0,513	0,340	0,177	0,050
3	0,250	0,000	0,000	0,000	0,000	0,000	3,226	2,624	2,181	1,834	1,548	1,299	1,071	0,856	0,648	0,448	0,269	0,128	0,041	0,005
4	0,200	0,000	0,000	0,000	0,000	3,853	3,067	2,508	2,080	1,729	1,424	1,146	0,883	0,636	0,412	0,230	0,104	0,035	0,007	0,000
5	0,167	0,000	0,000	0,000	4,843	3,742	2,984	2,428	1,987	1,611	1,271	0,955	0,663	0,411	0,219	0,096	0,033	0,008	0,001	0,000
6	0,143	0,000	0,000	6,653	4,783	3,673	2,917	2,348	1,879	1,465	1,085	0,739	0,446	0,232	0,100	0,035	0,009	0,002	0,000	0,000
7	0,125	0,000	0,000	6,547	4,719	3,617	2,850	2,254	1,748	1,290	0,875	0,526	0,271	0,117	0,042	0,012	0,003	0,000	0,000	0,000
8	0,111	0,000	0,000	6,474	4,670	3,563	2,775	2,144	1,591	1,092	0,663	0,345	0,151	0,055	0,016	0,004	0,001	0,000	0,000	0,000
9	0,100	0,000	10,000	6,422	4,627	3,506	2,688	2,012	1,410	0,882	0,472	0,211	0,079	0,024	0,006	0,001	0,000	0,000	0,000	0,000
10	0,091	0,000	9,941	6,381	4,585	3,443	2,586	1,858	1,212	0,679	0,317	0,122	0,039	0,011	0,002	0,000	0,000	0,000	0,000	0,000
11	0,083	0,000	9,879	6,346	4,541	3,371	2,467	1,684	1,006	0,499	0,202	0,068	0,019	0,004	0,001	0,000	0,000	0,000	0,000	0,000
12	0,077	0,000	9,831	6,313	4,495	3,289	2,331	1,494	0,806	0,351	0,125	0,037	0,009	0,002	0,000	0,000	0,000	0,000	0,000	0,000
13	0,071	0,000	9,793	6,282	4,444	3,195	2,176	1,293	0,623	0,239	0,075	0,019	0,004	0,001	0,000	0,000	0,000	0,000	0,000	0,000
14	0,067	0,000	9,761	6,251	4,387	3,088	2,005	1,091	0,467	0,158	0,044	0,010	0,002	0,000	0,000	0,000	0,000	0,000	0,000	0,000
15	0,063	0,000	9,734	6,219	4,323	2,967	1,819	0,897	0,340	0,102	0,025	0,005	0,001	0,000	0,000	0,000	0,000	0,000	0,000	0,000

Note: Parameter n corresponds to the number of players. This simulation is based on the assumption of bidder cost values uniformly distributed in the interval [0,1] as well as the reserve value. The parameter $\lambda^{prs}(0)$ in the second column explains situations in which the auction fails in equilibrium in the bidding without negotiation.
Source: Authors' calculations.

7. Conclusion: The problem of negotiation

Much has been discussed about whether or not it is desirable to keep the reserve price in a public tender secret. In the recent reform of the procurement auction law in Brazil, in addition to the secret value, a mechanism that also exists in the European Union was introduced, namely the ex post negotiation in order to guarantee the success of the bidding when the winning bid is still above the secret limit. In this case, the winner is asked if she would be willing to lower her value in order to match the maximum allowed.

Little has been studied about this innovative negotiation mechanism. The present work sought precisely to fill this gap, using, for that purpose, a procurement auction model of private, independent and identically distributed values. The derived equilibria and their simulations indicate some interesting results regarding the comparison of the different possible auction models.

First, including a publicly announced reserve price is better than not including it, in the sense that the government's net expected benefit is greater with the announced reserve price. There is, however, one caveat. By including the reserve price, the work is no longer contracted in some situations, which does not occur in the absence of a reserve price. Furthermore, the importance of competition is clear, since the increase in the number of participants practically eliminates the advantage of including the announced reservation price.

Second, including a secret reserve price is also better than not including any reserve price at all, with a similar caveat on not contracting the work.

Third, including a secret reserve price can be better or worse than announcing it publicly, being better for higher reserve values and worse for lower reserve values. This is a result similar to the one found in Silva (2011) in the context of interdependent values.

Fourth, if the reserve price reflects the social welfare generated by the auctioned work, including the possibility of negotiation in the model with a secret reserve price increases the players' bid, which causes a decrease in the benefit of including the secret reserve price. Therefore, the novel negotiation option tends to reduce the government benefits compared to the simple secret reserve price.

Fifth, unlike auction modeling, in which the reserve price is naturally defined as the value that the owner of the object attributes to the asset being auctioned, in the case of bidding there is some ambiguity in the definition of the reserve price. The natural adaptation of the auction formulation would be to identify the reserve price with the benefit that the work or service being bid generates to society. On the other hand, the current legislation is very clear in establishing that the reserve price must be the competitive value of construction of the work or provision of the service. There is a theoretical rationale for identifying the two concepts, competitive cost and social benefit; however, one can easily argue for the distinction between these two criteria. If the social benefit of a work or service is greater than its competitive cost, then its provision, even when the government pays its reserve value, would generate a strictly positive net benefit to the government. The present article shows that, in this case, there may be an additional advantage in the model in which negotiation is allowed, which may or may not compensate for the increased cost associated with this mechanism. In a nutshell, the use of the negotiation mechanism becomes advantageous when the reserve value increases as well as when the competition measured by the number of participants increases, situations in which the probability of not contracting the work increases when there is no negotiation. We believe that this is an important contribution to the literature that has the potential to guide policy makers in their decision of whether or not to use the negotiation mechanism.

The solution of the procurement auction model with secret reserve price and negotiation presents an implicit expression for the equilibrium, that is, the bid chosen by a bidder appears as the solution of an equation that depends on the primitive of the reserve value distribution function $H(r)$. The natural question that arises is under what conditions on H can we guarantee that this exists. This question is also presented here as a suggestion for future extensions.

Furthermore, considering the legislation in force that establishes how to estimate the value of a work, that is, the reserve value r, it can be assumed that the participants themselves are able to estimate that value quite accurately. In this case, although formally secret, it is possible that, in practice, the model with secret reserve prices performs in an equivalent way to the model with announced reserve price, as discussed in the introduction to this article. It may also be possible that some of the participants have access to this accurate estimate, either because they have better technical conditions to calculate it, or for other reasons associated with corruption, for example. In this case, the model with a secret reserve value can generate an asymmetry between the participants that can, ultimately, reduce the competitiveness of the event. The extension of the model to incorporate this asymmetry is also left here as a suggestion for future research.

The modeling developed in this article was based on the lowest cost procurement auction. This approach reflects current Brazilian legislation that provides for the payment of the lowest bid. Alternatively, one could consider second-lowest cost procurement auctions, where the winner is the one who makes the lowest bid, but he receives the second-lowest bid, that is, potentially receives more than his bid. This model is equivalent to Vickrey's Auction (Vickrey, 1961) and has very good properties, such as the fact that it is a revelation mechanism, that is, each participant has a (weakly) dominant strategy to bid its own cost. As we are following the paradigm of the model of private, independent, symmetric values with risk-neutral agents, it is likely that we can prove a Government Expenditure Equivalence Theorem, according to which the two formats, the lowest and the second lowest cost, are equivalent from the point of view of the expected expenditure of the government. This extension is also mentioned here as a suggestion for future research.

Also in addition to the initial motivation of this article, which is the analysis of the negotiation mechanism within the scope of the new public bidding law in Brazil, one could deepen the analysis of the government as a strategic agent, which could strategically choose the value of the reserve price, in order to minimize its expenditure. In the case of an auction with a confidential reserve value in which there is no negotiation, Elyakime et al. (1994) shows that the auctioneer's optimal strategy is to choose the value assigned to the object as the reserve price. However, it is unclear whether there could be another optimizing strategy when negotiation is possible. This strategic analysis is offered here as a suggestion for future research.

Finally, the study presented here fits the independent private values paradigm. Extension of the analysis to common or affiliated values as well as the possibility of collusion is also suggested for further exploration.

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Apêndice (We are sorry for the proofs in Portuguese)

Demonstração da Proposição 1

Buscamos um equilíbrio de Nash Bayesiano (ENB) simétrico, estritamente crescente diferenciável λ.

Suponha que os demais jogadores jogam segundo esse equilíbrio e o jogador i tem custo x_i e faz o lance l. Então i ganhará se e somente se:

$$l < \lambda(x_j),\ \forall j \neq i \quad \Leftrightarrow \quad l < \min_{j \neq i} \lambda(x_j)$$

Como λ é estritamente crescente, denotando por $m_1(x_{-i})$ a estatística de ordem 1 das $n-1$ observações $\{x_j\}_{j \neq i}$, i vence se e somente se:

$$l < \lambda\left(\min_{j \neq i}(x_j)\right) = \lambda(m_1(x_{-i})) \quad \Leftrightarrow \quad m_1(x_{-i}) > \lambda^{-1}(l)$$

Portanto, a probabilidade de vitória de i com lance l é:

$$\text{Prob}[m_1(x_{-i}) > \lambda^{-1}(l)] = 1 - \text{Prob}[m_1(x_{-i}) < \lambda^{-1}(l)] = 1 - G\big(\lambda^{-1}(l)\big)$$

Mas então, sua utilidade esperada (ínterim) é: $\big[1 - G\big(\lambda^{-1}(l)\big)\big](l - x_i)$

Se essa utilidade for uma função estritamente côncava, então a condição de primeira ordem (CPO) nos dará a melhor resposta de i às estratégias dos demais jogadores. Substituindo x_i por x por simplicidade,

$$\frac{\partial}{\partial l}\big[1 - G\big(\lambda^{-1}(l)\big)\big](l - x) = -g\big(\lambda^{-1}(l)\big)(\lambda^{-1})'(l)(l - x) + \big[1 - G\big(\lambda^{-1}(l)\big)\big] = 0$$

Na solução que se busca, $l = \lambda(x) \Leftrightarrow x = \lambda^{-1}(l)$. Portanto, podemos reescrever a condição acima como:

$$1 - G(x) = g(x)(\lambda^{-1})'\big(\lambda(x)\big)(\lambda(x) - x)$$

Note agora que $\lambda^{-1}\big(\lambda(x)\big) = x \Rightarrow (\lambda^{-1})'\big(\lambda(x)\big) = \big(\lambda'(x)\big)^{-1}$.

Portanto, a CPO se reduz a:

$$1 - G(x) = g(x)\big(\lambda'(x)\big)^{-1}(\lambda(x) - x)$$

$$[1 - G(x)]\lambda'(x) = g(x)(\lambda(x) - x) = g(x)\lambda(x) - g(x)x$$

$$g(x)x = -[1 - G(x)]\lambda'(x) + g(x)\lambda(x)$$

$$xg(x) = -\frac{d}{dx}[1 - G(x)]\lambda(x)$$

Mas então, pelo Teorema Fundamental do Cálculo,

$$[1 - G(\overline{\omega})]\lambda(\overline{\omega}) - [1 - G(x)]\lambda(x) = -\int_x^{\overline{\omega}} yg(y)dy$$

Mas $G(\overline{\omega}) = 1$. Portanto, a condição acima se reduz à expressão procurada:

$$\lambda(y) = \frac{1}{1 - G(x)}\int_x^{\overline{\omega}} yg(y)dy = E[m_1 \mid m_1 > x]$$

Ademais,

$$[yG(y)]' = G(y) + yg(y) \quad \Rightarrow \quad \overline{\omega} - xG(x) = \int_x^{\overline{\omega}} G(y)dy + \int_x^{\overline{\omega}} yg(y)dy$$

$$\Rightarrow \int_x^{\overline{\omega}} yg(y)dy = \overline{\omega} - xG(x) - \int_x^{\overline{\omega}} G(y)dy = (\overline{\omega} - x) + x\big(1 - G(x)\big) - \int_x^{\overline{\omega}} G(y)dy$$

$$= x\big(1 - G(x)\big) + \int_x^{\overline{\omega}} \big(1 - G(y)\big)dy$$

Portanto,

$$\lambda(x) = \frac{1}{1 - G(x)}\int_x^{\overline{\omega}} yg(y)dy = x + \int_x^{\overline{\omega}} \frac{1 - G(y)}{1 - G(x)}dy$$

Em particular, $\lambda(x) > x$ para todo $x > 0$.

Cálculos do Exemplo 1

Os valores são uniformemente distribuídos no intervalo [0,1]. Então:

$F(x) = x,\ \ G(x) = 1 - (1 - x)^{n-1},\ \ 1 - G(x) = (1 - x)^{n-1},$

$g(x) = (n - 1)(1 - x)^{n-2},\ \ yg(y) = (n - 1)y(1 - y)^{n-2}.$

Ademais, integrando por partes,

$$\int_x^1 yg(y)dy = yG(y)|_x^1 - \int_x^1 G(y)dy = 1 - xG(x) - \left[y + \frac{(1-y)^n}{n}\right]_x^1$$

$$= 1 - xG(x) - [1 - x] - \left[-\frac{(1-x)^n}{n}\right] = x\big(1 - G(x)\big) + \frac{(1-x)^n}{n}$$

$$= x(1-x)^{n-1} + \frac{(1-x)^n}{n} = (1-x)^{n-1}\left[\frac{nx + 1 - x}{n}\right] = (1-x)^{n-1}\left[\frac{(n-1)x + 1}{n}\right]$$

Portanto, o lance de um jogador de tipo x é:

$$\lambda^{spr}(x) = \frac{1}{1 - G(x)}\int_x^1 yg(y)dy = \frac{1}{(1-x)^{n-1}}\int_x^1 yg(y)dy = \frac{(n-1)x + 1}{n}$$

$$\lambda^{spr}(x) = \frac{(n-1)x + 1}{n} = \frac{n-1}{n}x + \frac{1}{n}$$

Demonstração da Proposição 2

Suponha que o jogador 1 incorra no custo x para fornecer a obra. Então, no equilíbrio simétrico encontrado ele vencerá se seu custo for o menor de todos. Isso ocorre com probabilidade $[1 - F(x)]^{n-1} = 1 - G(x)$.

Se vencer, o pagamento do governo será $\lambda^{spr}(x) = \frac{1}{1-G(x)}\int_x^{\overline{\omega}} yg(y)dy$.

Mas então, o pagamento esperado a um jogador de tipo x é:

$$m_1^{spr}(x) = Prob\big[x \leq x_j, j \neq 1\big]\lambda^{spr}(x) = [1 - G(x)]\frac{1}{1 - G(x)}\int_x^{\overline{\omega}} yg(y)dy$$

$$= \int_x^{\overline{\omega}} yg(y)dy$$

Portanto, o pagamento esperado para o jogador 1, considerando todos os seus possíveis tipos é:

$$m_1^{spr} = \int_{\underline{\omega}}^{\overline{\omega}}\int_x^{\overline{\omega}} yg(y)dy\, f(x)dx = \int_{\underline{\omega}}^{\overline{\omega}}\int_{\underline{\omega}}^{y} f(x)dx\, yg(y)dy = \int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy$$

Mas então, o pagamento esperado do governo, considerando todos os jogadores é:

$$m^{spr} = n\int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy$$

Cálculos do Exemplo 2

Os valores são uniformemente distribuídos no intervalo [0,1]. Então:

$F(x) = x,\;\; G(x) = 1-(1-x)^{n-1},\;\; 1-G(x) = (1-x)^{n-1},$

$g(x) = (n-1)(1-x)^{n-2},\;\; yg(y) = (n-1)y(1-y)^{n-2}.$

Então, $F(y)yg(y) = y(yg(y))$.

Sejam $a(y) = y$ e $b'(y) = yg(y)$.

Então $a'(y) = 1$ e $b(y) = \int_0^y zg(z)\,dz$

Sejam $c(z) = z$ e $d'(z) = g(z)$.

Então $c'(z) = 1$ e $d(z) = G(z) = 1-(1-z)^{n-1}$.

Portanto, $b(y) = z[1-(1-z)^{n-1}]|_0^y - \int_0^y G(z)\,dz =$

$= y[1-(1-y)^{n-1}] - \int_0^y [1-(1-z)^{n-1}]\,dz$

$= y - y(1-y)^{n-1} - \left[z + \frac{(1-z)^n}{n}\right]\Big|_0^y = y - y(1-y)^{n-1} - \left[y + \frac{(1-y)^n}{n} - \frac{1}{n}\right]$

$b(y) = \frac{1}{n} - y(1-y)^{n-1} - \frac{(1-y)^n}{n}$

Mas então:

$$\int_0^1 F(y)yg(y)dy = [yb(y)]|_0^1 - \int_0^1 b(y)dy$$

$$= \frac{1}{n} - \frac{1}{n} - \frac{1}{n(n+1)}(1-y)^{n+1}|_0^1 + \int_0^1 y(1-y)^{n-1}\,dy$$

$$= \frac{1}{n(n+1)} + \int_0^1 y(1-y)^{n-1}\,dy$$

Sejam $h(y) = y$ e $i'(y) = (1-y)^{n-1}$.

Então $h'(y) = 1$ e $i(y) = -\frac{1}{n}(1-y)^n$.

Então, $\int_0^1 y(1-y)^{n-1}\,dy = -\frac{1}{n}(1-y)^n y\Big|_0^1 + \frac{1}{n}\int_0^1 (1-y)^n dy$

$$= 0 - \frac{1}{n(n+1)}(1-y)^{n+1}\Big|_0^1 = \frac{1}{n(n+1)}$$

Portanto, $\int_0^1 F(y)yg(y)dy = \frac{2}{n(n+1)}$.

Mas então,

$$m^{spr} = n\int_0^1 F(y)yg(y)dy = \frac{2}{n+1}$$

Demonstração da Proposição 3

Se um jogador tiver custo $x > r$, então não tem interesse em executar a obra e fazer um lance igual a seu custo é uma melhor resposta que lhe garante retorno zero. O pagamento esperado a esse jogador será zero.

Se um jogador tiver valor $x \leq r$, então, está claro que uma melhor resposta sua envolverá um lance $l = \lambda(x) \leq r$. Ademais, para que seu lance seja ótimo é necessário que $l = \lambda(x) \geq x$.

Tomando o limite com $x \to r$, com valores abaixo de r conclui-se que em qualquer equilíbrio de Nash bayesiano simétrico contínuo, $\lambda(r) = r$. Essa será a “condição de contorno”.

Buscamos um ENB λ simétrico, estritamente crescente e diferenciável (para custos menores ou iguais a r). Se um jogador i tiver custo $x_i < r$ e fizer um lance l, então esse jogador ganhará se e somente se: $l < \lambda(x_j),\ \forall j \neq i.$ Ou ainda, $l < \min_{j\neq i} \lambda(x_j)$.

Como λ é estritamente crescente, i vence se e somente se:

$$l < \lambda\left(\min_{j\neq i}(x_j)\right) = \lambda\big(m_1(x_{-i})\big) \quad \Leftrightarrow \quad m_1(x_{-i}) > \lambda^{-1}(l)$$

Portanto, a probabilidade de vitória de i com lance l é: $1 - G\big(\lambda^{-1}(l)\big)$.

Mas então, escrevendo x em vez de x_i, sua utilidade esperada (ínterim) é:

$$\left(1 - G\big(\lambda^{-1}(l)\big)\right)(l - x)$$

Portanto, o jogador resolve o mesmo problema que na situação sem preço de reserva, com a única alteração sendo a condição de contorno $\lambda(r) = r$. Resolvendo o problema da mesma forma que antes chegamos à seguinte equação diferencial:

$$xg(x) = -\frac{d}{dx}[1 - G(x)]\lambda(x)$$

Pelo Teorema Fundamental do Cálculo, agora aplicado entre x e r,

$$[1 - G(x)]\lambda(x) - [1 - G(r)]\lambda(r) = \int_x^r yg(y)dy$$

Como, $\lambda(r) = r$, a condição acima se reduz a:

$$\lambda(x) = r\frac{1 - G(r)}{1 - G(x)} + \frac{1}{1 - G(x)}\int_x^r yg(y)dy = E[\min\{r, m_1\} \mid m_1 > x]$$

Cálculos do Exemplo 3

Os valores são uniformemente distribuídos no intervalo [0,1]. Então:

$F(x) = x,\ \ G(x) = 1 - (1-x)^{n-1},\ \ 1 - G(x) = (1-x)^{n-1},$

$g(x) = (n-1)(1-x)^{n-2},\ \ yg(y) = (n-1)y(1-y)^{n-2}.$

integrando por partes, para $x \leq r$,

$$\int_x^r yg(y)dy = yG(y)|_x^r - \int_x^r G(y)dy = rG(r) - xG(x) - \left[y + \frac{(1-y)^n}{n}\right]_x^r$$

$$= rG(r) - xG(x) - \left[r + \frac{(1-r)^n}{n} - x - \frac{(1-x)^n}{n}\right]$$

$$= x\big(1 - G(x)\big) - r\big(1 - G(r)\big) + \frac{1}{n}[(1-x)^n - (1-r)^n]$$

Portanto,

$$\frac{1}{1-G(x)}\int_x^r yg(y)dy = x - r\frac{\big(1-G(r)\big)}{1-G(x)} + \frac{1}{n}\left[\frac{(1-x)^n}{1-G(x)} - \frac{(1-r)^n}{1-G(x)}\right]$$

Substituindo,

$$\frac{1}{1-G(x)}\int_x^r yg(y)dy = x - r\frac{(1-r)^{n-1}}{(1-x)^{n-1}} + \frac{1}{n}\left[(1-x) - \frac{(1-r)^n}{(1-x)^{n-1}}\right]$$

$$\lambda^{pra}(x) = r\frac{1-G(r)}{1-G(x)} + \frac{1}{1-G(x)}\int_x^r yg(y)dy = x + \frac{1}{n}\left[(1-x) - \frac{(1-r)^n}{(1-x)^{n-1}}\right]$$

$$\lambda^{pra}(x) = x + \frac{1}{n}\left[(1-x) - \frac{(1-r)^n}{(1-x)^{n-1}}\right]$$

$$\lambda^{pra}(x) = \frac{n-1}{n}x + \frac{1}{n} - \frac{1}{n}\frac{(1-r)^n}{(1-x)^{n-1}}$$

Demonstração da Proposição 4

Suponha que o jogador 1 incorra no custo $x \leq r$ para fornecer a obra. Então, no equilíbrio simétrico encontrado ele vencerá se seu custo for o menor de todos. Isso ocorre com probabilidade $[1 - F(x)]^{n-1} = 1 - G(x)$.

Se vencer, o pagamento do governo será:

$$\lambda^{pra}(x) = r\frac{1-G(r)}{1-G(x)} + \frac{1}{1-G(x)}\int_x^r yg(y)dy$$

Mas então, o pagamento esperado a um jogador de tipo $x \leq r$ é:

$$m_1^{pra}(x) = Prob\big[x \leq x_j, j \neq 1\big]\lambda^{pra}(x) = [1 - G(x)]\lambda^{pra}(x)$$

$$m_1^{pra}(x) = r(1 - G(r)) + \int_x^r yg(y)dy$$

Portanto, o pagamento esperado para o jogador 1, considerando todos os seus possíveis tipos é:

$$m_1^{pra} = \int_{\underline{\omega}}^{r} m_1^{pra}(x) f(x) dx$$

$$= \int_{\underline{\omega}}^{r} r\big(1 - G(r)\big) f(x) dx + \int_{\underline{\omega}}^{r} \int_{x}^{r} y g(y) dy\, f(x) dx$$

Mas,

$$\int_{\underline{\omega}}^{r} \int_{x}^{r} y g(y) dy\, f(x) dx = \int_{\underline{\omega}}^{r} \int_{\underline{\omega}}^{y} f(x) dx\, y g(y) dy = \int_{\underline{\omega}}^{r} \big[F(y) - F\big(\underline{\omega}\big)\big] y g(y) dy$$

Portanto,

$$m_1^{pra} = r\big(1 - G(r)\big)F(r) + \int_{\underline{\omega}}^{r} F(y) y g(y) dy$$

Mas então, o pagamento esperado do governo, considerando todos os jogadores é:

$$m^{pra}(r) = nr\big(1 - G(r)\big)F(r) + n \int_{\underline{\omega}}^{r} F(y) y g(y) dy$$

Demonstração do Corolário às Proposições 3 e 4

Para $x > r$, $\lambda^{pra}(x) = r\frac{1-G(r)}{1-G(x)} + \frac{1}{1-G(x)}\int_x^r y g(y) dy = \lambda(x; r)$.

Portanto, para $r < 1$,

$$\big(1 - G(x)\big)\frac{\partial \lambda(x; r)}{\partial r} = 1 - G(r) - r g(y) + r g(r) = 1 - G(r) > 0$$

Ademais, o pagamento esperado do governo é:

$$m^{pra}(r) = nr\big(1 - G(r)\big)F(r) + n \int_{\underline{\omega}}^{r} F(y) y g(y) dy$$

Portanto, para $r < 1$,

$$\frac{1}{n}\frac{\partial m^{pra}(r)}{\partial r} = \big[\big(1 - G(r)\big)F(r) - r g(r) F(r) + r(1 - G(r) f(r)\big] + F(r) r g(r)$$

$$= \big(1 - G(r)\big)F(r) + r(1 - G(r) f(r) = \big(1 - G(r)\big)\big(F(r) + r f(r)\big) > 0$$

Cálculos do Exemplo 4

Os valores são uniformemente distribuídos no intervalo [0,1]. Então: $\underline{\omega} = 0$ e:

$F(x) = x,\ \ G(x) = 1 - (1 - x)^{n-1},\ \ 1 - G(x) = (1 - x)^{n-1},$

$g(x) = (n - 1)(1 - x)^{n-2},\ \ y g(y) = (n - 1) y (1 - y)^{n-2}.$

Então, $nr\big(1 - G(r)\big)F(r) = nr(1 - r)^{n-1} r = nr^2(1 - r)^{n-1}$

Ademais, $F(y)yg(y) = y\big(yg(y)\big) = y^2g(y)$.

Sejam $a(y) = y^2$ e $b'(y) = g(y)$.

Então $a'(y) = 2y$ e $b(y) = \int_0^y g(z)\,dz = G(y)$

Portanto,

$$\int_0^r F(y)yg(y)dy = y^2G(y)|_0^r - 2\int_0^r yG(y)dy = r^2G(r) - 2\int_0^r yG(y)dy$$

Sejam $c(z) = z$ e $d'(z) = G(z)$.

Então $c'(z) = 1$ e $d(z) = \int_0^z G(w)dw = \int_0^z [1-(1-w)^{n-1}]dw$.

Ou ainda, $d(z) = w|_0^z + \frac{1}{n}(1-w)^n|_0^z = z + \frac{1}{n}(1-z)^n - \frac{1}{n}$.

Portanto,

$$\int_0^r yG(y)dy = z\left[z + \frac{1}{n}(1-z)^n - \frac{1}{n}\right]_0^r - \int_0^r \left[z + \frac{1}{n}(1-z)^n - \frac{1}{n}\right]dz$$

$$= r\left[r + \frac{1}{n}(1-r)^n - \frac{1}{n}\right] - \left[\frac{z^2}{2} - \frac{1}{n(n+1)}(1-z)^{n+1} - \frac{1}{n}z\right]_0^r$$

$$= r\left[r + \frac{1}{n}(1-r)^n - \frac{1}{n}\right] - \left[\frac{r^2}{2} - \frac{1}{n(n+1)}(1-r)^{n+1} - \frac{1}{n}r + \frac{1}{n(n+1)}\right]$$

$$= \frac{r^2}{2} + \frac{1}{n}r(1-r)^n - \frac{1}{n}r + \frac{1}{n}\frac{1}{n+1}(1-r)^{n+1} + \frac{1}{n}r - \frac{1}{n}\frac{1}{n+1}$$

$$= \frac{r^2}{2} - \frac{1}{n(n+1)} + \frac{1}{n}(1-r)^n\left[r + \frac{1}{n+1}(1-r)\right]$$

$$= \frac{r^2}{2} - \frac{1}{n(n+1)} + \frac{1}{n(n+1)}(1-r)^n[(n+1)r + (1-r)]$$

$$= \frac{r^2}{2} - \frac{1}{n(n+1)} + \frac{1}{n(n+1)}(1-r)^n[nr+1]$$

Logo,

$$\int_0^r F(y)yg(y)dy = r^2G(r) - 2\int_0^r yG(y)dy$$

$$= r^2[1-(1-r)^{n-1}] - r^2 + \frac{2}{n(n+1)} - \frac{2}{n(n+1)}(1-r)^n[nr+1]$$

$$= +\frac{2}{n(n+1)} - r^2(1-r)^{n-1} - \frac{2}{n(n+1)}(1-r)^n[nr+1]$$

Portanto,

$$n\int_0^r F(y)yg(y)dy = \frac{2}{n+1} - nr^2(1-r)^{n-1} - \frac{2}{n+1}(1-r)^n[nr+1]$$

Como $nr\left(1-F(r)\right)^{n-1}F(r) = nr^2(1-r)^{n-1}$, conclui-se que:

$$m^{pra} = \frac{2}{n+1} - \frac{2}{n+1}(1-r)^n[nr+1]$$

Ou ainda,

$$m^{pra}(r) = \frac{2}{n+1}[1-(1-r)^n(nr+1)]$$

Demonstração da Proposição 5

$$m^{pra}(r) = nr\left(1-G(r)\right)F(r) + n\int_{\underline{\omega}}^{r} F(y)yg(y)dy$$

$$m^{spr} = n\int_{\underline{\omega}}^{\overline{\omega}} F(y)yg(y)dy = n\int_{\underline{\omega}}^{r} F(y)yg(y)dy + n\int_{r}^{\overline{\omega}} F(y)yg(y)dy$$

Portanto, $m^{pra}(r) < m^{spr} \Leftrightarrow \int_r^{\overline{\omega}} F(y)yg(y)dy > r\left(1-G(r)\right)F(r)$.

Mas, para todo $y \in [r, \overline{\omega}]$, $F(y)y > F(r)r \Rightarrow \int_r^{\overline{\omega}} F(y)yg(y)dy > \int_r^{\overline{\omega}} F(r)rg(y)dy = r\left(1-G(r)\right)F(r)$.

Demonstração da Proposição 6

Buscamos um equilíbrio de Nash Bayesiano (ENB) simétrico, estritamente crescente e diferenciável λ.

Suponha que os demais jogadores jogam segundo esse equilíbrio e o jogador i tem custo x_i e faz o lance l. Então i ganhará e executará a obra se duas condições forem satisfeitas:

(i) $l < \lambda(x_j),\ \forall j \neq i \quad \Leftrightarrow \quad l < \min_{j\neq i} \lambda(x_j)$.

(ii) $l \leq r$.

Como λ é estritamente crescente, i vence se e somente se:

$$l < \lambda\left(\min_{j\neq i}(x_j)\right) = \lambda\left(m_1(x_{-i})\right) \quad \Leftrightarrow \quad m_1(x_{-i}) > \lambda^{-1}(l)$$

Portanto, a probabilidade de i fazer o menor lance l é:

$$\text{Prob}[m_1(x_{-i}) > \lambda^{-1}(l)] = 1 - \text{Prob}[m_1(x_{-i}) < \lambda^{-1}(l)] = 1 - G\left(\lambda^{-1}(l)\right)$$

Ademais, a probabilidade de que o lance l esteja abaixo do valor de reserva é:

$$\text{Prob}[r \geq l] = 1 - \text{Prob}[r < l] = 1 - H(l)$$

Mas então, sua utilidade esperada (ínterim) é: $\left[1 - G\left(\lambda^{-1}(l)\right)\right][1-H(l)](l - x_i)$

Se essa utilidade for uma função estritamente côncava, então a CPO nos dará a melhor resposta de i às estratégias dos demais jogadores. Substituindo x_i por x por simplicidade,

$$\frac{\partial}{\partial l}[1-G(\lambda^{-1}(l))][1-H(l)](l-x)$$
$$=-g(\lambda^{-1}(l))(\lambda^{-1})'(l)[1-H(l)](l-x)+[1-G(\lambda^{-1}(l))][1-H(l)]$$
$$-[1-G(\lambda^{-1}(l))]h(l)(l-x)=0$$

Na solução que se busca, $l=\lambda(x)\Leftrightarrow x=\lambda^{-1}(l)$. Portanto, podemos reescrever a condição acima como:

$$[1-G(x)][1-H(\lambda(x))]$$
$$-g(x)(\lambda^{-1})'(\lambda(x))[1-H(\lambda(x))](\lambda(x)-x)-[1-G(x)]h(\lambda(x))(\lambda(x)-x)=0$$

Note agora que $\lambda^{-1}(\lambda(x))=x\Rightarrow(\lambda^{-1})'(\lambda(x))=(\lambda'(x))^{-1}$.

Portanto, a CPO se reduz a:

$$[1-G(x)][1-H(\lambda(x))]$$
$$-g(x)(\lambda'(x))^{-1}[1-H(\lambda(x))](\lambda(x)-x)-[1-G(x)]h(\lambda(x))(\lambda(x)-x)=0$$

ou ainda,

$$[1-G(x)][1-H(\lambda(x))](\lambda'(x)-1)$$
$$-g(x)[1-H(\lambda(x))](\lambda(x)-x)-[1-G(x)]h(\lambda(x))\lambda'(x)(\lambda(x)-x)=$$
$$=-[1-G(x)][1-H(\lambda(x))]$$

Equivalentemente,

$$\frac{\partial}{\partial x}[1-G(x)][1-H(\lambda(x))](\lambda(x)-x)=-[1-G(x)][1-H(\lambda(x))]$$

Pelo Teorema Fundamental do Cálculo, integrando de x a $\overline{\omega}$, e lembrando que $G(\overline{\omega})=1$, temos:

$$[1-G(x)][1-H(\lambda(x))](\lambda(x)-x)=\int_x^{\overline{\omega}}[1-G(y)][1-H(\lambda(y))]dy$$

Donde,

$$\lambda(x)=x+\frac{1}{1-G(x)}\int_x^{\overline{\omega}}\frac{1-H(\lambda(y))}{1-H(\lambda(x))}[1-G(y)]dy$$

ou ainda,

$$\lambda(x)=x+\frac{1}{[1-G(x)][1-H(\lambda(x)]}\int_x^{\overline{\omega}}[1-H(\lambda(y))][1-G(y)]dy$$

Demonstração do Corolário à Proposição 6

De fato, temos, por um lado,

$$\lambda^{spr}(x) = x + \frac{1}{1-G(x)}\int_x^{\overline{\omega}} [1-G(y)]dy$$

e, por outro,

$$\lambda^{prs}(x) = x + \frac{1}{1-G(x)}\int_x^{\overline{\omega}} \frac{1-H\big(\lambda(y)\big)}{1-H\big(\lambda(x)\big)}[1-G(y)]dy$$

Mas, para todo $y \in (x, \overline{\omega})$, $H\big(\lambda(y)\big) > H\big(\lambda(x)\big)$ se H for estritamente crescente.

Então,

$$1-H\big(\lambda(y)\big) < 1-H\big(\lambda(x)\big) \Rightarrow \frac{1-H\big(\lambda(y)\big)}{1-H\big(\lambda(x)\big)} < 1 \;\Rightarrow\; \lambda^{prs}(x) < \lambda^{spr}(x)$$

Cálculos do Exemplo 5

Os valores são uniformemente distribuídos no intervalo [0,1], inclusive quanto ao valor de reserva r. Então:

$F(x) = x,\; G(x) = 1-(1-x)^{n-1},\; 1-G(x) = (1-x)^{n-1},$

$g(x) = (n-1)(1-x)^{n-2},\; yg(y) = (n-1)y(1-y)^{n-2},\; H(r) = r,\; h(r) = 1.$

Então[14], para $x < 1$,

$$\lambda^{prs}(x) = x + \frac{1}{(1-x)^{n-1}}\int_x^1 \frac{1-\lambda(y)}{1-\lambda(x)}[1-y]^{n-1}dy$$

Busquemos um equilíbrio de Nash bayesiano linear, i.e., $\lambda^{prs}(x) = \alpha x + \beta$. Então a equação acima pode ser reescrita como:

$$\lambda^{prs}(x) = x + \frac{1}{(1-x)^{n-1}(1-\alpha x-\beta)}\int_x^1 (1-\alpha y-\beta)[1-y]^{n-1}dy$$

Sejam $\quad a(y) = 1-\alpha y-\beta \;\Rightarrow\; a'(y) = -\alpha$

$\qquad b'(y) = [1-y]^{n-1} \;\Rightarrow\; b(y) = -\frac{[1-y]^n}{n}$

Então,

$$\int_x^1 (1-\alpha y-\beta)[1-y]^{n-1}dy = -[1-\alpha y-\beta]\left[\frac{(1-y)^n}{n}\right]_x^1 - \frac{\alpha}{n}\int_x^1 [1-y]^n dy$$

$$= [1-\alpha x-\beta]\left[\frac{(1-x)^n}{n}\right] + \frac{\alpha}{n}\left[\frac{(1-y)^{n+1}}{n+1}\right]_x^1$$

[14] Naturalmente, $\lambda(1) = 1$.

$$= [1 - \alpha x - \beta]\left[\frac{(1-x)^n}{n}\right] - \frac{\alpha}{n}\left[\frac{(1-x)^{n+1}}{n+1}\right]$$

$$= \frac{(1-x)^n}{n}\left[1 - \alpha x - \beta - \frac{\alpha}{n+1}(1-x)\right]$$

$$= \frac{(1-x)^n}{n(n+1)}[(n+1)(1-\alpha x-\beta) - \alpha(1-x)]$$

$$= \frac{(1-x)^n}{n(n+1)}[n(1-\alpha x-\beta) + 1 - \alpha - \beta]$$

Mas então,

$$\lambda^{prs}(x) = \alpha x + \beta = x + \frac{(1-x)^n}{n(n+1)}\left[\frac{n(1-\alpha x-\beta)+1-\alpha-\beta}{(1-x)^{n-1}(1-\alpha x-\beta)}\right]$$

$$= x + \frac{1-x}{n(n+1)}\left[\frac{n(1-\alpha x-\beta)+1-\alpha-\beta}{(1-\alpha x-\beta)}\right]$$

$$= x + \frac{1-x}{n(n+1)}\left[n + \frac{1-\alpha-\beta}{1-\alpha x-\beta}\right]$$

$$\lambda^{prs}(x) = \alpha x + \beta = x + \frac{1-x}{n+1} + \left(\frac{1-\alpha-\beta}{1-\alpha x-\beta}\right)\frac{1-x}{n(n+1)}$$

$$\lambda^{prs}(x) = \alpha x + \beta = \frac{n}{n+1}x + \frac{1}{n+1} + \frac{1-\alpha-\beta}{n(n+1)}\frac{1-x}{1-\alpha x-\beta}$$

Cuja solução nos leva a: $\alpha = \frac{n}{n+1}$, $\beta = \frac{1}{n+1}$, ou seja,

$$\lambda^{prs}(x) = \frac{n}{n+1}x + \frac{1}{n+1}$$

Demonstração da Proposição 7

Suponha que o jogador 1 incorra no custo x para fornecer a obra. Então, no equilíbrio simétrico encontrado ele vencerá se seu custo for o menor de todos e, além disso, seu lance for menor que o preço de reserva.

Seu custo será menor que os demais com probabilidade $[1 - F(x)]^{n-1} = 1 - G(x)$.

Ademais, seu lance $\lambda(x)$ será menor que o valor de reserva r se $\lambda(x) \leq r \Leftrightarrow x \leq \lambda^{-1}(r) =: s$.

Note que se $\lambda(\underline{\omega}) > r$, então nunca teremos $\lambda(x) \leq r$, de forma que a obra não será contratada e o pagamento do governo será zero.

Por outro lado, se $\lambda(\underline{\omega}) \leq r$, então a obra será contratada se $\lambda(x) \leq r \Leftrightarrow x \leq \lambda^{-1}(r) =: s$. Nesse caso, o governo pagará:

$$[1-G(x)]\lambda(x) = x[1-G(x)] + \int_x^{\overline{\omega}} \frac{1-H\big(\lambda(y)\big)}{1-H\big(\lambda(x)\big)}[1-G(y)]dy$$

Ou ainda,

$$[1-G(x)]\lambda(x) = x[1-G(x)] + \frac{1}{1-H\big(\lambda(x)\big)}\int_x^{\overline{\omega}} \big[1-H\big(\lambda(y)\big)\big][1-G(y)]dy$$

Mas então, se $s = \lambda^{-1}(r) \geq \underline{\omega}$, o pagamento esperado ao jogador 1, considerando todos os seus possíveis tipos é:

$$\int_{\underline{\omega}}^{s} [1-G(x)]\lambda(x)f(x)dx$$

$$= \int_{\underline{\omega}}^{s} x[1-G(x)]f(x)dx + \int_{\underline{\omega}}^{s}\int_x^{\overline{\omega}} \big[1-H\big(\lambda(y)\big)\big][1-G(y)]dy \frac{1}{1-H\big(\lambda(x)\big)} f(x)dx$$

Mas então podemos concluir que o pagamento esperado do governo, considerando todos os jogadores será: $m^{prs}(r) = 0$ se $\lambda\big(\underline{\omega}\big) > r$ e, se $\lambda\big(\underline{\omega}\big) \leq r$, então o pagamento esperado será:

$$m^{prs}(r) =$$

$$n\int_{\underline{\omega}}^{\lambda^{-1}(r)} x[1-G(x)]f(x)dx$$

$$+ n\int_{\underline{\omega}}^{\lambda^{-1}(r)}\int_x^{\overline{\omega}} \big[1-H\big(\lambda(y)\big)\big][1-G(y)]dy \frac{1}{1-H\big(\lambda(x)\big)} f(x)dx$$

Cálculos do Exemplo 6

Os valores são uniformemente distribuídos no intervalo [0,1], com a solução linear. Então: $\underline{\omega} = 0, \overline{\omega} = 1$ e:

$F(x) = x,\ \ G(x) = 1-(1-x)^{n-1},\ \ 1-G(x) = (1-x)^{n-1},\ \ H(x) = x$

Ademais,

$$\lambda^{prs}(x) = \frac{n}{n+1}x + \frac{1}{n+1} \ \Rightarrow\ 1-H\big(\lambda(y)\big) = \frac{n}{n+1}(1-y)$$

E,

$$\frac{n}{n+1}x + \frac{1}{n+1} = r \ \Leftrightarrow\ \frac{n}{n+1}x = r - \frac{1}{n+1}$$

$$x = \frac{n+1}{n}r - \frac{1}{n} = \lambda^{-1}(r) =: s$$

Sejam

$$A(s) = \int_0^s x[1-G(x)]f(x)dx$$

$$B(s) = \int_0^s \int_x^1 [1 - H(\lambda(y))][1 - G(y)]dy \frac{1}{1 - H(\lambda(x))} f(x)dx$$

Então,

$$m^{prs}(r) = nA(\lambda^{-1}(r)) + nB(\lambda^{-1}(r))$$

Calculemos,

$$A(s) = \int_0^s x(1-x)^{n-1} dx$$

Sejam $a(x) = x$ e $b'(x) = (1-x)^{n-1}$.

Então $a'(x) = 1$ e $b(x) = -\frac{(1-x)^n}{n}$.

Portanto,

$$A(s) = -x\left[\frac{(1-x)^n}{n}\right]_0^s + \int_0^s \frac{(1-x)^n}{n} dx$$

$$= -\frac{1}{n}s(1-s)^n - \frac{1}{n}\left[\frac{(1-x)^{n+1}}{n+1}\right]_0^s$$

$$= \frac{1}{n}\left[-s(1-s)^n - \frac{1}{n+1}\ [(1-s)^{n+1} - 1]\right]$$

$$= \frac{1}{n}\frac{1}{n+1}[1 - (n+1)s(1-s)^n - (1-s)^{n+1}]$$

Mas,

$$(n+1)s(1-s)^n + (1-s)^{n+1} = (1-s)^n[ns + s + 1 - s]$$
$$= (1-s)^n(1+ns)$$

Logo,

$$A(s) = \frac{1}{n}\frac{1}{n+1}[1 - (1-s)^n(1+ns)]$$

Mas,

$$s = \lambda^{-1}(r) = \frac{n+1}{n}r - \frac{1}{n} \Rightarrow -s = \frac{1}{n} - \frac{n+1}{n}r$$

$$1 - s = \frac{n+1}{n}(1-r) \Rightarrow (1-s)^n = \left(\frac{n+1}{n}\right)^n (1-r)^n$$

$$(1+ns) = (1 + (n+1)r - 1) = (n+1)r$$

Portanto,

$$A(s) = \frac{1}{n}\frac{1}{n+1}\left[1 - \left(\frac{n+1}{n}\right)^n (1-r)^n(n+1)r\right]$$

$$A(s) = \frac{1}{n}\frac{1}{n+1}\left[1 - (n+1)\left(\frac{n+1}{n}\right)^n r(1-r)^n\right]$$

$$A(s) = \frac{1}{n}\frac{1}{n+1}\frac{(n+1)^{n+1}}{n^n}\left[\frac{n^n}{(n+1)^{n+1}} - r(1-r)^n\right]$$

$$A(s) = \frac{(n+1)^n}{n^{n+1}}\left[\frac{n^n}{(n+1)^{n+1}} - r(1-r)^n\right]$$

E,

$$nA(s) = \left(\frac{n+1}{n}\right)^n \left[\frac{n^n}{(n+1)^{n+1}} - r(1-r)^n\right]$$

Ou ainda,

$$nA(s) = \frac{1}{n+1} - \left(\frac{n+1}{n}\right)^n r(1-r)^n$$

Calculemos agora,

$$B(s) = \int_0^s \int_x^1 [1 - H(\lambda(y))][1 - G(y)]dy \frac{1}{1 - H(\lambda(x))} f(x)dx$$

$$\int_x^1 [1 - H(\lambda(y))][1 - G(y)]dy = \int_x^1 \frac{n}{n+1}(1-y)(1-y)^{n-1}dy$$

$$= \frac{n}{n+1}\int_x^1 (1-y)^n dy = -\frac{n}{n+1}\frac{1}{n+1}\ (1-y)^{n+1}|_x^1 = \frac{n}{(n+1)^2}(1-x)^{n+1}$$

$$B(s) = \int_0^s \frac{n}{(n+1)^2}(1-x)^{n+1}\frac{n+1}{n}\frac{1}{1-x}dx = \frac{1}{n+1}\int_0^s (1-x)^n dx$$

$$= -\frac{1}{n+1}\frac{1}{n+1}(1-x)^{n+1}|_0^s = -\frac{1}{(n+1)^2}[(1-s)^{n+1} - 1]$$

$$= \frac{1}{(n+1)^2}[1 - (1-s)^{n+1}]$$

Lembrando que

$$1 - s = \frac{n+1}{n}(1-r) \Rightarrow (1-s)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1}(1-r)^{n+1}$$

$$B(s) = \frac{1}{(n+1)^2}\left[1 - \left(\frac{n+1}{n}\right)^{n+1}(1-r)^{n+1}\right]$$

E,

$$nB(s) = \frac{n}{(n+1)^2}\left[1 - \left(\frac{n+1}{n}\right)^{n+1}(1-r)^{n+1}\right]$$

Ou ainda,

$$nB(s) = \frac{n}{(n+1)^2} - \frac{(n+1)^{n-1}}{n^n}(1-r)^{n+1}$$

Portanto,

$$m^{prs}(r) = nA\big(\lambda^{-1}(r)\big) + nB(\lambda^{-1}(r))$$

$$= \frac{1}{n+1} - \left(\frac{n+1}{n}\right)^n r(1-r)^n + \frac{n}{(n+1)^2} - \frac{(n+1)^{n-1}}{n^n}(1-r)^{n+1}$$

$$= \frac{1}{n+1}\left[1 + \frac{n}{n+1}\right] - \frac{(n+1)^{n-1}}{n^n}[(n+1)r + (1-r)](1-r)^n$$

$$= \frac{1}{n+1}\frac{n+1+n}{n+1} - \frac{(n+1)^{n-1}}{n^n}[nr + r + 1 - r](1-r)^n$$

$$m^{prs}(r) = \frac{2n+1}{(n+1)^2} - \frac{(n+1)^{n-1}}{n^n}[nr+1](1-r)^n$$

Demonstração da Proposição 8

Buscamos um equilíbrio de Nash Bayesiano (ENB) simétrico, estritamente crescente diferenciável λ.

Suponha que os demais jogadores jogam segundo esse equilíbrio e o jogador i tem custo x_i e faz o lance l. Então seu lance será o lance vencedor se e somente se:

$$l < \lambda(x_j),\ \forall j \neq i \quad \Leftrightarrow \quad l < \min_{j \neq i} \lambda(x_j)$$

Como λ é estritamente crescente, i vence se e somente se:

$$l < \lambda\left(\min_{j \neq i}(x_j)\right) = \lambda\big(m_1(x_{-i})\big) \quad \Leftrightarrow \quad m_1(x_{-i}) > \lambda^{-1}(l)$$

Portanto, a probabilidade de i fazer o menor lance l é:

$$\text{Prob}[m_1(x_{-i}) > \lambda^{-1}(l)] = 1 - \text{Prob}[m_1(x_{-i}) < \lambda^{-1}(l)] = 1 - G\big(\lambda^{-1}(l)\big)$$

Se esse jogador fizer o menor lance, três situações podem ocorrer.

(i) $l \leq r$. Nesse caso, esse jogador é contratado e recebe o pagamento l. Seu *payoff* resultante é $l - x_i$.

(ii) $r < l$ mas $x \leq r$. Então o jogador aceita fornecer a obra pelo valor r durante a fase de negociação. Portanto, esse jogador é selecionado para executar a obra e recebe por ela o pagamento r. Seu *payoff* resultante é $r - x_i$.

(iii) $r < l$ e $r < x$. Nesse caso não interessa ao jogador o pagamento r, a licitação fracassa e o licitante que fez o menor lance tem *payoff* 0.

Portanto, a utilidade esperada ínterim desse jogador é:

$$\big[1 - G\big(\lambda^{-1}(l)\big)\big]\left[\int_{\underline{\omega}}^{x} 0h(r)dr + \int_{x}^{l}(r-x)\,h(r)dr + \int_{l}^{\overline{\omega}}(l-x)\,h(r)dr\right]$$

$$= \left[1 - G\left(\lambda^{-1}(l)\right)\right]\left[\int_x^l (r-x)\,h(r)dr + \int_l^{\overline{\omega}} (l-x)\,h(r)dr\right]$$

Mas,

$$\int_x^l (r-x)\,h(r)dr = \int_x^l r\,h(r)dr - x[H(l) - H(x)]$$

Sejam $a(r) = r$ e $b'(r) = h(r)$.

Então $a'(r) = 1$ e $b(r) = H(r)$.

Portanto,

$$\int_x^l r\,h(r)dr = rH(r)|_x^l - \int_x^l H(r)dr = lH(l) - xH(x) - \mathcal{H}(l) + \mathcal{H}(x)$$

Logo,

$$\int_x^l (r-x)\,h(r)dr = (l-x)H(l) - \mathcal{H}(l) + \mathcal{H}(x)$$

Ademais,

$$\int_l^{\overline{\omega}} (l-x)\,h(r)dr = (l-x)\left(H(\overline{\omega}) - H(l)\right) = (l-x)\left(1 - H(l)\right)$$

Mas então,

$$\int_x^l (r-x)\,h(r)dr + \int_l^{\overline{\omega}} (l-x)\,h(r)dr = (l-x) - \mathcal{H}(l) + \mathcal{H}(x)$$

Portanto, a utilidade esperada do licitante pode ser reescrita como:

$$\left[1 - G\left(\lambda^{-1}(l)\right)\right][\,(l-x) - \mathcal{H}(l) + \mathcal{H}(x)]$$

Se essa utilidade for uma função estritamente côncava, então a CPO nos dará a melhor resposta de i às estratégias dos demais jogadores. Substituindo x_i por x por simplicidade,

$$\frac{\partial}{\partial l}\left[1 - G\left(\lambda^{-1}(l)\right)\right][\,(l-x) - \mathcal{H}(l) + \mathcal{H}(x)]$$

$$= -g\left(\lambda^{-1}(l)\right)(\lambda^{-1})'(l)[\,(l-x) - \mathcal{H}(l) + \mathcal{H}(x)] + \left[1 - G\left(\lambda^{-1}(l)\right)\right]\left(1 - H(l)\right) = 0$$

Na solução que se busca, $l = \lambda(x) \Leftrightarrow x = \lambda^{-1}(l)$. Portanto, podemos reescrever a condição acima como:

$$-\left[\lambda(x) - x - \mathcal{H}\left(\lambda(x)\right) + \mathcal{H}(x)\right]g(x)(\lambda^{-1})'\left(\lambda(x)\right) + [1 - G(x)]\left(1 - H\left(\lambda(x)\right)\right) = 0$$

Note agora que $\lambda^{-1}\left(\lambda(x)\right) = x \Rightarrow (\lambda^{-1})'\left(\lambda(x)\right) = \left(\lambda'(x)\right)^{-1}$.

Portanto, a CPO se reduz a:

$$-\left[\lambda(x) - x - \mathcal{H}\left(\lambda(x)\right) + \mathcal{H}(x)\right]g(x) + [1 - G(x)]\left(1 - H\left(\lambda(x)\right)\right)\lambda'(x) = 0$$

Note agora que:

$$\frac{d}{dx}\left[\lambda(x) - x - \mathcal{H}\big(\lambda(x)\big) + \mathcal{H}(x)\right] = \lambda'(x) - 1 - H\big(\lambda(x)\big)\lambda'(x) + H(x)$$

$$= \Big(1 - H\big(\lambda(x)\big)\Big)\lambda'(x) - \big(1 - H(x)\big)$$

Mas,

$$[1 - G(x)]\Big(1 - H\big(\lambda(x)\big)\Big)\lambda'(x)$$

$$= [1 - G(x)]\left[\Big(1 - H\big(\lambda(x)\big)\Big)\lambda'(x) - \big(1 - H(x)\big)\right] + [1 - H(x)][1 - G(x)]$$

Portanto, a CPO pode ser reescrita como:

$$[1 - G(x)]\left[\Big(1 - H\big(\lambda(x)\big)\Big)\lambda'(x) - \big(1 - H(x)\big)\right] + [-g(x)]\left[\lambda(x) - x - \mathcal{H}\big(\lambda(x)\big) + \mathcal{H}(x)\right]$$

$$= -[1 - H(x)][1 - G(x)]$$

Ou ainda,

$$\frac{d}{dx}[1 - G(x)]\left[\lambda(x) - x - \mathcal{H}\big(\lambda(x)\big) + \mathcal{H}(x)\right] = -[1 - H(x)][1 - G(x)]$$

Mas então, pelo Teorema Fundamental do Cálculo,

$$[1 - G(x)]\left[\lambda(x) - x - \mathcal{H}\big(\lambda(x)\big) + \mathcal{H}(x)\right]_x^{\overline{\omega}} = -\int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)]dy$$

Como $G(\overline{\omega}) = 1$, a condição acima se reduz à expressão procurada:

$$\lambda(x) - x - \mathcal{H}\big(\lambda(x)\big) + \mathcal{H}(x) = \frac{1}{1 - G(x)}\int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)]dy$$

Portanto,

$$\lambda(x) = x + \mathcal{H}\big(\lambda(x)\big) - \mathcal{H}(x) + \frac{1}{1 - G(x)}\int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)]dy$$

Cálculos do Exemplo 7

Os valores são uniformemente distribuídos no intervalo [0,1], inclusive quanto ao valor de reserva r. Então:

$F(x) = x,\ \ G(x) = 1 - (1 - x)^{n-1},\ \ 1 - G(x) = (1 - x)^{n-1},$

$g(x) = (n - 1)(1 - x)^{n-2},\ \ yg(y) = (n - 1)y(1 - y)^{n-2},\ \ H(r) = r,\ \ h(r) = 1.$

Então, $\mathcal{H}(x) = \frac{x^2}{2}$.

Portanto,

$$\lambda(x) - \frac{\lambda^2(x)}{2} = x - \frac{x^2}{2} + \frac{1}{(1 - x)^{n-1}}\int_x^1 [1 - y][(1 - y)^{n-1}]dy$$

$$= \frac{1}{2}(2x - x^2) - \frac{1}{(1-x)^{n-1}} \frac{(1-y)^{n+1}}{(n+1)}\bigg|_x^1 = \frac{1}{2}(2x - x^2) + \frac{(1-x)^2}{n+1}$$

$$= \frac{1}{2}\frac{1}{n+1}[2(n+1)x - (n+1)x^2 + (2 - 4x + 2x^2)]$$

$$= \frac{1}{2}\frac{1}{n+1}[2(n-1)x - (n-1)x^2 + 2] \qquad (1)$$

Busquemos um equilíbrio de Nash bayesiano linear, i.e., $\lambda^{prs}(x) = \alpha x + \beta$. Então a equação acima pode ser reescrita como:

$$\lambda(x) - \frac{\lambda^2(x)}{2} = \alpha x + \beta - \frac{\alpha^2}{2}x^2 - \alpha\beta x - \frac{\beta^2}{2} = \alpha(1-\beta)x - \frac{\alpha^2}{2}x^2 + \frac{1}{2}\beta(2-\beta) \quad (2)$$

Comparando as expressões (1) e (2) chegamos a:

$$\alpha = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}, \quad \beta = 1 - \alpha = 1 - \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$$

Portanto, a solução procurada é:

$$\lambda^{psn}(x) = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} x + \left(1 - \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

Demonstração da Proposição 9

Suponha que o jogador 1 incorra no custo x para fornecer a obra. Então, no equilíbrio simétrico encontrado ele vencerá se seu custo for o menor de todos e, além disso:

(i) seu lance for menor que o preço de reserva ou ainda que,

(ii) tendo feito um lance maior que o preço de reserva, seu custo seja menor que esse preço.

Seu custo será menor que os demais com probabilidade $[1 - F(x)]^{n-1} = 1 - G(x)$.

Seu lance $\lambda(x)$ será menor que o valor de reserva r se: $\lambda(x) \leq r \Leftrightarrow x \leq \lambda^{-1}(r) =: s$.

Ademais, seu lance $\lambda(x)$ será maior que r, mas seu custo será menor que r se:

$$x \leq r < \lambda(x) \Leftrightarrow \lambda^{-1}(x) < \lambda^{-1}(r) =: s < x$$

Ou ainda, $\lambda(x) > r \Leftrightarrow x > \lambda^{-1}(r) =: s$ mas $x \leq r$.

Consideremos dois casos.

Caso 1: $\lambda(\underline{\omega}) > r$

Nesse caso todos os lances serão acima do valor de reserva r. Portanto, a obra somente será contratada se $x \leq r$, de forma que:

Se $x > r$ a licitação fracassa e o pagamento do governo é zero.

Se $x \leq r$ a obra é contratada e o pagamento do governo é:

$$[1 - G(x)]r$$

Portanto, o pagamento esperado a esse jogador é:

$$r\int_0^r [1 - G(x)]f(x)dx$$

Logo, o pagamento esperado considerando todos os jogadores nesse caso é:

$$m^{psn}(r) = nr\int_0^r [1 - G(x)]f(x)dx$$

Caso 2: $\lambda(\underline{\omega}) \leq r$

Subcaso 2.1: $\lambda(x) \leq r \Leftrightarrow x \leq \lambda^{-1}(r)$

Então, o governo pagará:

$$[1 - G(x)]\lambda(x)$$

Subcaso 2.2: $(\lambda(x) > r) \quad \wedge \quad \left(x \leq r \Leftrightarrow \lambda^{-1}(x) \leq \lambda^{-1}(r) =: s(< x)\right)$

Então, o governo pagará:

$$[1 - G(x)]r$$

Subcaso 2.3: $(\lambda(x) > r) \quad \wedge \quad (x > r \Leftrightarrow \lambda^{-1}(x) > \lambda^{-1}(r) =: s(< x))$

Nesse caso a obra não é contratada: o vencedor desiste de executar a obra na etapa de negociação, pois seu custo é maior que o valor de reserva.

Mas então, o pagamento esperado ao jogador 1, considerando todos os seus possíveis tipos é:

$$\int_{\underline{\omega}}^{s} [1 - G(x)]\lambda(x)f(x)dx + r\int_s^r [1 - G(x)]f(x)dx$$

Seja $A(s) = \int_{\underline{\omega}}^{s} [1 - G(x)]\lambda(x)f(x)dx$

Como

$$\lambda(x) = x + \mathcal{H}\big(\lambda(x)\big) - \mathcal{H}(x) + \frac{1}{1 - G(x)}\int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)]dy$$

$$A(s) = \int_{\underline{\omega}}^{s} [1 - G(x)]\lambda(x)f(x)dx =$$

$$= \int_{\underline{\omega}}^{s} x[1 - G(x)]f(x)dx + \int_{\underline{\omega}}^{s}\int_x^{\overline{\omega}} [1 - H(y)][1 - G(y)]dy\, f(x)dx$$

$$+\int_{\underline{\omega}}^{s}[\mathcal{H}(\lambda(x))-\mathcal{H}(x)][1-G(x)]f(x)dx$$

Mas então podemos concluir que o pagamento esperado do governo, considerando todos os jogadores neste segundo caso é:

$$m^{psn}(r)=$$

$$n\int_{\underline{\omega}}^{\lambda^{-1}(r)}x[1-G(x)]f(x)dx+\ n\int_{\underline{\omega}}^{\lambda^{-1}(r)}\int_{x}^{\overline{\omega}}[1-H(y)][1-G(y)]dy\,f(x)dx$$

$$+n\int_{\underline{\omega}}^{\lambda^{-1}(r)}[\mathcal{H}(\lambda(x))-\mathcal{H}(x)][1-G(x)]f(x)dx+nr\int_{\lambda^{-1}(r)}^{r}[1-G(x)]f(x)dx$$

Cálculos do Exemplo 8

Os valores são uniformemente distribuídos no intervalo [0,1], com a solução linear. Então: $\underline{\omega}=0,\overline{\omega}=1$ e:

$F(x)=x,\ \ G(x)=1-(1-x)^{n-1},\ \ 1-G(x)=(1-x)^{n-1},\ \ H(x)=x$

Então, $\mathcal{H}(x)=\frac{x^2}{2}$ e

$$\lambda^{psn}(x)=\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}x+\left(1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

$$\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}x+\left(1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)=r\Leftrightarrow\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}x=r-\left(1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

$$x=\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}r-\left(\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}-1\right)=\lambda^{-1}(r)=:s$$

Analisemos separadamente os dois possíveis casos.

Caso 1: $\lambda(\underline{\omega})>r$

$$m^{psn}(r)=nr\int_{0}^{r}[1-G(x)]f(x)dx=nr\int_{0}^{r}[1-x]^{n-1}dx=r(-1)(1-x)^{n}|_{0}^{r}$$

$$m^{psn}(r)=r(-(1-r)^{n}+1)=r(1-(1-r)^{n})$$

Caso 2: $\lambda(\underline{\omega})\leq r$

Sejam

$$A(s)=\int_{0}^{s}x[1-G(x)]f(x)dx$$

$$B(s) = \int_0^s \int_x^1 [1 - H(y)][1 - G(y)]dy\, f(x)dx$$

$$C(s) = \int_0^s [\mathcal{H}\big(\lambda(x)\big) - \mathcal{H}(x)][1 - G(x)]f(x)dx$$

$$D(s) = \; r\int_s^r [1 - G(x)]f(x)dx$$

Então,

$$m^{psn}(r) = nA\big(\lambda^{-1}(r)\big) + nB\big(\lambda^{-1}(r)\big) + nC\big(\lambda^{-1}(r)\big) + nD\big(\lambda^{-1}(r)\big)$$

Primeiramente calculemos $A(s)$. Conforme visto anteriormente (no exemplo após a Proposição 7),

$$A(s) = \frac{1}{n}\frac{1}{n+1}[1 - (1-s)^n(1+ns)]$$

Mas $s = \lambda^{-1}(r) = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}} r + 1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}} = 1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r)$, portanto,

$$1 - s = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r) \Rightarrow (1-s)^n = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n$$

E $\;\; 1 + ns = 1 + n\left(1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r)\right)$

Logo,

$$[1 - (1-s)^n(1+ns)] = \left[1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n\left[1 + n\left(1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r)\right)\right]\right]$$

Portanto,

$$nA(s) = \frac{1}{n+1}\left[1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n\left[1 + n\left(1 - \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r)\right)\right]\right]$$

Calculemos agora $B(s)$:

$$B(s) = \int_0^s \int_x^1 [1-y][1-y]^{n-1}dy\,dx = \int_0^s \int_x^1 (1-y)^n dy\,dx$$

$$= -\frac{1}{n+1}\int_0^s (1-y)^{n+1}|_x^1 dx = \frac{1}{n+1}\int_0^s (1-x)^{n+1}dx$$

$$\frac{1}{n+1}\frac{1}{n+2}(-1)(1-x)^{n+2}|_0^s = -\frac{1}{n+1}\frac{1}{n+2}[(1-s)^{n+2} - 1]$$

$$nB(s) = \frac{n}{n+1}\frac{1}{n+2}[1-(1-s)^{n+2}]$$

Mas,

$$1-s = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r) \Rightarrow (1-s)^{n+2} = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}(n+2)}(1-r)^{n+2}$$

Logo,

$$nB(s) = \frac{n}{n+1}\frac{1}{n+2}\left[1-\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}(n+2)}(1-r)^{n+2}\right]$$

$$nB(s) = \frac{n}{n+1}\frac{1}{n+2} - \frac{n}{n+1}\frac{1}{n+2}\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}(n+2)}(1-r)^{n+2}$$

$$nB(s) = \frac{n}{n+1}\frac{1}{n+2} - \frac{n}{n+2}\frac{(n+1)^{\frac{n}{2}}}{(n-1)^{\frac{1}{2}(n+2)}}(1-r)^{n+2}$$

Ou ainda,

$$nB(s) = \frac{n}{n+1}\frac{1}{n+2}\left[1-\frac{n+1}{n-1}\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^{n+2}\right]$$

Continuando,

$$C(s) = \int_{\underline{\omega}}^{s}[\mathcal{H}\big(\lambda(x)\big)-\mathcal{H}(x)][1-G(x)]f(x)dx$$

$$C(s) = \int_{\underline{\omega}}^{s}\left[\frac{\lambda(x)^2}{2}-\frac{x^2}{2}\right][1-x]^{n-1}dx$$

$$\lambda^{psn}(x) = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}x + \left(1-\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\right)$$

Sejam $\alpha = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$ e $\beta = 1-\alpha$. Então: $\lambda(x) = \alpha x + \beta$

$$\mathcal{H}(x) = \frac{x^2}{2}$$

$$\mathcal{H}(\lambda(x)) = \frac{\alpha^2x^2 + 2\alpha\beta x + \beta^2}{2}$$

$$\frac{\lambda(x)^2}{2} - \frac{x^2}{2} = \frac{1}{2}[(\alpha^2-1)x^2 + 2\alpha\beta x + \beta^2]$$

Portanto,

$$C(s) = \int_{\underline{\omega}}^{s} \left[\frac{\lambda(x)^2}{2} - \frac{x^2}{2}\right][1-x]^{n-1}dx$$

$$= \frac{1}{2}\int_0^s [(\alpha^2-1)x^2 + 2\alpha\beta x + \beta^2][1-x]^{n-1}dx$$

$$a(x) = (\alpha^2-1)x^2 + 2\alpha\beta x + \beta^2 \quad \Rightarrow \quad a'(x) = 2(\alpha^2-1)x + 2\alpha\beta$$

$$b'(x) = [1-x]^{n-1} \qquad \Rightarrow \quad b(x) = -\tfrac{1}{n}[1-x]^n$$

$$\int_0^s [(\alpha^2-1)x^2 + 2\alpha\beta x + \beta^2][1-x]^{n-1}dx$$

$$= [a(x)b(x)]|_0^s + \frac{1}{n}\int_0^s [2(\alpha^2-1)x + 2\alpha\beta][1-x]^n dx$$

$$= -a(s)\frac{1}{n}[1-s]^n + \frac{1}{n}\beta^2 + \frac{1}{n}\int_0^s [2(\alpha^2-1)x + 2\alpha\beta][1-x]^n dx$$

$$c(x) = 2(\alpha^2-1)x + 2\alpha\beta \quad \Rightarrow \quad c'(x) = 2(\alpha^2-1)$$

$$d'(x) = [1-x]^n \qquad \Rightarrow \quad d(x) = -\tfrac{1}{n+1}[1-x]^{n+1}$$

$$\int_0^s [2(\alpha^2-1)x + 2\alpha\beta][1-x]^n dx$$

$$= [c(x)d(x)]|_0^s + 2(\alpha^2-1)\frac{1}{n+1}\int_0^s [1-x]^{n+1}dx$$

$$= -c(s)\frac{1}{n+1}[1-s]^{n+1} + 2\alpha\beta\frac{1}{n+1} + 2(\alpha^2-1)\frac{1}{n+1}\frac{1}{n+2}[-(1-s)^{n+2}+1]$$

$$= \frac{1}{n+1}\left[\frac{1}{n+2}2(\alpha^2-1) + 2\alpha\beta\right] - \frac{1}{n+1}\left[c(s)[1-s]^{n+1} + \frac{1}{n+2}2(\alpha^2-1)(1-s)^{n+2}\right]$$

$$= \frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2-1) + 2\alpha\beta\right] - \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2-1)(1-s)\right][1-s]^{n+1}$$

Então,

$$\int_0^s [(\alpha^2-1)x^2 + 2\alpha\beta x + \beta^2][1-x]^{n-1}dx$$

$$= -a(s)\frac{1}{n}[1-s]^n + \frac{1}{n}\beta^2$$

$$+\frac{1}{n}\left[\frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2-1) + 2\alpha\beta\right] - \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2-1)(1-s)\right][1-s]^{n+1}\right]$$

$$= \frac{1}{n}\left[\beta^2 + \frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2-1) + 2\alpha\beta\right]\right]$$

$$-\frac{1}{n}\left[a(s)[1-s]^n + \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2-1)(1-s)\right][1-s]^{n+1}\right]$$

$$= \frac{1}{n}\left[\beta^2 + \frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2 - 1) + 2\alpha\beta\right]\right]$$

$$-\frac{1}{n}\left[a(s) + \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2 - 1)[1-s]\right][1-s]\right][1-s]^n$$

Finalmente,

$$nC(s) = \frac{1}{2}\left[\beta^2 + \frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2 - 1) + 2\alpha\beta\right]\right]$$

$$-\frac{1}{2}\left[a(s) + \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2 - 1)[1-s]\right][1-s]\right][1-s]^n$$

Lembrando que $1 - s = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}[1-r]$,

$$nC(s) = \frac{1}{2}\left[\beta^2 + \frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2 - 1) + 2\alpha\beta\right]\right]$$

$$-\frac{1}{2}\left[a(s) + \frac{1}{n+1}\left[c(s) + \frac{2}{n+2}(\alpha^2 - 1)\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}[1-r]\right]\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}[1-r]\right]\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n$$

Continuando,

$$D(s) = r\int_s^r [1 - G(x)]f(x)dx = r\int_s^r [1-x]^{n-1}dx = -\frac{r}{n}[1-x]^n|_s^r$$

$$= -\frac{r}{n}[[1-r]^n - [1-s]^n] = \frac{r}{n}[[1-s]^n - [1-r]^n]$$

Como $[1-s]^n = \left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n$,

$$D(s) = \frac{1}{n}\left[\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n} - 1\right]r[1-r]^n$$

Portanto,

$$nD(s) = \left[\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n} - 1\right]r[1-r]^n$$

Mas então, o pagamento esperado procurado é:

$$m^{psn}(r) = nA\big(\lambda^{-1}(r)\big) + nB\big(\lambda^{-1}(r)\big) + nC\big(\lambda^{-1}(r)\big) + nD\big(\lambda^{-1}(r)\big)$$

$$m^{psn}(r) = \frac{1}{n+1} - \left(\frac{n+1}{n}\right)^n r(1-r)^n$$

$$+\frac{n}{n+1}\frac{1}{n+2}-\frac{n}{n+2}\frac{(n+1)^{\frac{n}{2}}}{(n-1)^{\frac{1}{2}(n+2)}}(1-r)^{n+2}$$

$$+\frac{1}{2}\left[\beta^2+\frac{1}{n+1}\left[\frac{1}{n+2}2(\alpha^2-1)+2\alpha\beta\right]\right]$$

$$-\frac{1}{2}\left[a(s)+\frac{1}{n+1}\left[c(s)+\frac{2}{n+2}(\alpha^2-1)\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}[1-r]\right]\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}[1-r]\right]\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}(1-r)^n$$

$$+\left[\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}n}-1\right]r[1-r]^n$$

em que:

$$\alpha=\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$$

$$\beta=1-\alpha$$

$$s=\frac{1}{\alpha}r-\left(\frac{1}{\alpha}-1\right)=1-\frac{1}{\alpha}(1-r)$$

$$a(s)=(\alpha^2-1)s^2+2\alpha\beta s+\beta^2$$

$$c(s)=2(\alpha^2-1)s+2\alpha\beta$$

Alternativamente, podemos escrever: $\alpha=\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}, \beta=1-\alpha$

$$s=\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}r-\left(\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}-1\right)=\alpha^{-1}r+1-\alpha^{-1}\Rightarrow 1-s=\alpha^{-1}(1-r)$$

$$=\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}(1-r)$$

Portanto, o pagamento esperado do leiloeiro nesse caso pode ser escrito como:

$m^{psn}(r)=nA(s)+nB(s)+nC(s)+nD(s)$ em que:

$$nA(s)=\frac{1}{n+1}[1-(1-s)^n(1+ns)]$$

$$nB(s)=\frac{n}{n+1}\frac{1}{n+2}[1-(1-s)^{n+2}]$$

$$nC(s)=\frac{1}{2}\left[\beta^2+\frac{1}{n+1}\left[\frac{2}{n+2}(\alpha^2-1)+2\alpha\beta\right]\right]$$

$$-\frac{1}{2}\left[a(s)+\frac{1}{n+1}\left[c(s)+\frac{2}{n+2}(\alpha^2-1)[1-s]\right][1-s]\right][1-s]^n$$

$$nD(s) = r[[1-s]^n - [1-r]^n]$$

$s = 1 - \alpha^{-1}(1-r)$, $\alpha = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$, $\beta = 1-\alpha$, $a(s) = (\alpha^2-1)s^2 + 2\alpha\beta s + \beta^2$

$c(s) = 2(\alpha^2-1)s + 2\alpha\beta$.

Demonstração da Proposição 10.

Suponha que $b > r$. O retorno líquido esperado para o governo na licitação com negociação é:

$$b\left(1-\left(1-F(r)\right)^n\right) - m^{psn}(r)$$

Por outro lado, o retorno líquido esperado na licitação sem negociação é:

$$b\left(1-\left(1-F\left((\lambda^{prs})^{-1}(r)\right)\right)^n\right) - m^{prs}(r)$$

Portanto, será (estritamente) vantajoso para o governo usar o instrumento da negociação se:

$$b\left(1-\left(1-F(r)\right)^n\right) - m^{psn}(r) > b\left(1-\left(1-F\left((\lambda^{prs})^{-1}(r)\right)\right)^n\right) - m^{prs}(r)$$

$$\Leftrightarrow\ b\left(\left(1-F\left((\lambda^{prs})^{-1}(r)\right)\right)^n - \left(1-F(r)\right)^n\right) > m^{psn}(r) - m^{prs}(r)$$

Como λ^{prs} é estritamente crescente e $\lambda^{prs}(r) > r$, então $(\lambda^{prs})^{-1}(r) < r$.

Portanto, $\left(1-F\left((\lambda^{prs})^{-1}(r)\right)\right)^n > \left(1-F(r)\right)^n$ e a condição acima se escreve como:

$$b > \frac{m^{psn}(r) - m^{prs}(r)}{\left(1-F\left((\lambda^{prs})^{-1}(r)\right)\right)^n - \left(1-F(r)\right)^n}$$